

Temporal breakdown and Borel resummation in the complex Langevin method

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We reexamine the Parisi-Klauder conjecture for complex $e^{i\theta/2}\phi^4$ measures with a Wick rotation angle $0 \leq \theta/2 \leq \pi/2$ interpolating between Euclidean and Lorentzian signature. Our main result is that the asymptotics for short stochastic times t encapsulates information also about the equilibrium aspects. The moments evaluated with the complex measure and with the real measure defined by the stochastic Langevin equation have the same $t \rightarrow 0$ asymptotic expansion which is shown to be Borel summable. The Borel transform correctly reproduces the time dependent moments of the complex measure for all t , including their $t \rightarrow \infty$ equilibrium values. On the other hand the results of a direct numerical simulation of the Langevin moments are found to disagree from the ‘correct’ result for t larger than a finite t_c . The breakdown time t_c increases powerlike for decreasing strength of the noise’s imaginary part but cannot be excluded to be finite for purely real noise. To ascertain the discrepancy we also compute the real equilibrium distribution for complex noise explicitly and verify that its moments differ from those obtained with the complex measure.

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1. Introduction

The complex Langevin method [1, 2] is arguably the best candidate framework to define and compute Lorentzian signature functional integrals beyond series expansions. In brief it aims at replacing functional averages of some quantity \mathcal{O} with the e^{iS} integrand (‘complex measure’) by a limit of averages computed with a real measure on a ‘doubled’ configuration space, see [3] for a review. In the case of a one-component scalar field theory

$$\frac{\int \mathcal{D}\phi \mathcal{O}(\phi) e^{iS(\phi)}}{\int \mathcal{D}\phi e^{iS(\phi)}} = \lim_{\theta \rightarrow \pi^-} \lim_{t \rightarrow \infty} \frac{\int \mathcal{D}\phi_r \mathcal{D}\phi_i \mathcal{O}(\phi_r + i\phi_i) R_{t,\theta}(\phi_r, \phi_i)}{\int \mathcal{D}\phi_r \mathcal{D}\phi_i R_{t,\theta}(\phi_r, \phi_i)}, \quad (1.1)$$

where $R_{t,\theta}$ is a real positive measure for all t and θ such that $R_t(\phi_r, \phi_i) \rightarrow \delta(\phi_r)\delta(\phi_i)$ for $t \rightarrow 0$. The angle $0 \leq \theta < \pi$ is related to the phase of the Wick rotation and normalized such that $\theta = 0$ and π correspond to Euclidean and Lorentzian signature, respectively. The right hand side can be evaluated numerically for fixed t and θ by a two component version of the usual Langevin method. That is, a pair of stochastic differential equations driven by white noise b_r, b_i is solved for each instance of b_r, b_i and $\mathcal{O}(\phi_r^b + i\phi_i^b)$ is evaluated on the solution $\phi_r^b(t), \phi_i^b(t)$ at t , after which the ensemble average $\overline{\mathcal{O}(\phi_r^b + i\phi_i^b)}$ yields the ratio in (1.1) for fixed t, θ .

The numerical implementation of the Langevin method is seductively simple. The results of unguided numerical experiments are however often inconclusive: the large time limit of the averages entering the right hand side of (1.1) may fail to exist or may converge to the ‘wrong’ answer, in simple cases where the left hand side of (1.1) can be evaluated by other means, see [12] for a recent critical discussion. There is also a considerable body of mathematical work on the subject which unfortunately does not seem to cover the situations directly relevant to (1.1) and its numerical implementation. What is lacking is a theoretical understanding of the domain of validity of the method: if it fails, why does it fail, and conversely in what circumstances can one be assured that the right hand side of (1.1) indeed evaluates the left hand side.

Although the interest in the complex Langevin method comes mostly from field theory gradients are not at the core of the issue. In line with earlier investigations [5, 8, 9, 10, 11] we will therefore focus on the zero dimensional case and specifically on the paradigmatic case of a ϕ^4 interaction. To appreciate the origin of the angle θ and the simplifications made compared to a $1+d$ dimensional ϕ^4 theory we quickly run through the main steps of the Wick rotation in a lattice formulation. We start from the discretized Minkowski space action on a cylinder $T \times L^d$, with different lattice spacings a_0, a in the temporal and the spatial directions. This gives

$$S_M = \frac{1}{2} \frac{a^{d-1}}{a_0} \sum_n (\Delta_0 \phi_n)^2 - \frac{1}{2} a^{d-3} a_0 \sum_{n,i} (\Delta_i \phi_n)^2 - \frac{m^2}{2} a^{d-1} a_0 \sum_n \phi_n^2 - \frac{\lambda}{4} a^{d-1} a_0 \sum_n \phi_n^4, \quad (1.2)$$

where n labels the site, $T = N_0 a_0$, $L = Na$, and $\Delta_0 \phi_n := \phi_{n+\hat{0}} - \phi_n$, $\Delta_i \phi_n = \phi_{n+\hat{i}} - \phi_n$, $i = 1, 2, \dots, d$. Replacing $a_0 \mapsto -ie^{-i\theta/2}a_0$ and reverting to lattice units $a_0 = a = 1$ the partition function becomes

$$Z_\theta = \int \prod_n d\phi_n \exp(-S_\theta),$$

$$S_\theta := \frac{1}{2}e^{-i\theta/2} \sum_n (\Delta_0 \phi_n)^2 + e^{i\theta/2} \sum_n \left[\frac{1}{2}(\Delta_i \phi_n)^2 + \frac{m^2}{2}\phi_n^2 + \frac{\lambda}{4}\phi_n^4 \right]. \quad (1.3)$$

This multidimensional integral over the real variables ϕ_n is absolutely convergent for all $-\pi < \theta < \pi$. Evaluating the two point function based on (1.3) for $\lambda = 0$ and $\theta = \pi - \epsilon$ with ϵ a positive infinitesimal, one recovers the discretized Feynman propagator. This suggests that suitable distributional limits of the Z_θ based correlators with $\theta \rightarrow \pi - \epsilon$ would in principle define the Lorentzian signature lattice theory. Setting aside the subtle distributional aspects Z_θ is for $\theta \neq 0$ also not suited for evaluations based on numerical stochastic approaches (Monte Carlo, Langevin, etc) as the exponent is dominated by the imaginary parts, inducing an intolerably low signal to noise ratio. The Euclidean signature version corresponds to $\theta = 0$ and circumvents both awkward features, at the expense of a more indirect recovery of the Lorentzian signature amplitudes eventually aimed at. The gradients in (1.3) are not central to the problem. By discarding them one obtains a zero dimensional system whose stochastic quantization resembles $1 + 0$ dimensional quantum field theory (i.e. quantum mechanics) with the stochastic time providing the added dimension. In the following we write $S(q)$ for a (in general complex valued) polynomial action for the real variable q which carries a θ dependence induced by (1.3) interpolating between Lorentzian ($\theta = \pi$) and Euclidean signature ($\theta = 0$).

Associated with the complex action $S(q)$ is the complex Fokker-Planck equation

$$\frac{\partial}{\partial t} \rho_t(q) = \mathbf{P} \rho_t, \quad \mathbf{P} = \partial_q (\partial_q + \partial_q S), \quad (1.4)$$

in the stochastic time t with initial condition $\rho_{t=0}(q) = \rho_0(q)$. Here \mathbf{P} is the transpose of the usual Langevin operator $\mathbf{L} = \partial_q^2 - \partial_q S \partial_q$ and formally $\rho_t(x) = \exp(t\mathbf{P})\rho_0$. One assumes that for suitable initial data ρ_t exists and reproduces the complex Boltzmann factor in the limit $t \rightarrow \infty$, i.e. $\lim_{t \rightarrow \infty} \rho_t = e^{-S}$. For an ‘observable’ $\mathcal{O}(q)$ the average wrt the complex measure ρ_t is defined by

$$\langle \mathcal{O} \rangle_{\rho_t} = \frac{\int dq \mathcal{O}(q) \rho_t(q)}{\int dq \rho_t(q)}. \quad (1.5)$$

On the other hand the real measure relevant for the right hand side of (1.1) is defined by the following real Fokker-Planck equation

$$\frac{\partial}{\partial t} R_t(x, y) = \mathbb{P} R_t(x, y),$$

$$\mathbb{P} = \partial_x (A_R \partial_x - F_x) + \partial_y (A_R \partial_y - F_y), \quad A_R - A_I = 1,$$

$$F_x = -\text{Re}[\partial_x S(x + iy)], \quad F_y = -\text{Im}[\partial_x S(x + iy)], \quad (1.6)$$

where the forces satisfy $\partial_y F_x + \partial_x F_y = 0$ and $A_R = A_I + 1$, $A_I \geq 0$, reflects a variant of the fluctuation-dissipation theorem. The initial conditions are $R_0(x, y) = \rho_0(x)\delta(y)$, and again one has assume that $R_t = \exp(t\mathbb{P})R_0$ is well defined and has a limit for $t \rightarrow \infty$. For analytic observables (depending on $x + iy$ only) one considers the averages

$$\langle \mathcal{O} \rangle_{R_t} = \frac{\int dx dy \mathcal{O}(x + iy) R_t(x, y)}{\int dx dy R_t(x, y)}. \quad (1.7)$$

The Parisi-Klauder conjecture [1, 2] states that under ‘suitable subsidiary conditions’ both averages coincide:

$$\langle \mathcal{O} \rangle_{\rho_t} \stackrel{?}{=} \langle \mathcal{O} \rangle_{R_t}, \quad \text{for all } t \geq 0. \quad (1.8)$$

Then $\lim_{t \rightarrow \infty} \langle \mathcal{O} \rangle_{\rho_t} = \lim_{t \rightarrow \infty} \langle \mathcal{O} \rangle_{R_t}$ should follow and the simulation of the two-component Langevin equation can for large t be used to compute the expectation values with the complex Boltzmann factor $e^{-S(q)}$ as in (1.1).

For definiteness we focus on the case of quartic action $S(q) = \alpha e^{i\theta/2} q^4$, $0 < \alpha$, $0 \leq \theta < \pi$, where the moments $\langle x^p \rangle_{\rho_t}$ and $\langle (x + iy)^p \rangle_{R_t}$, with $p \in \mathbb{N}$, fully characterize the underlying measures. We show in Section 3 that both sets of moments have identical $t \rightarrow 0$ asymptotic expansions of the form

$$\langle x^p \rangle_{\rho_t} \sim \sum_{n \geq p/2 \pmod{2}} c_{p,n} (-4\alpha e^{i\theta/2})^{\frac{n-p/2}{2}} \frac{(2t)^n}{n!} \sim \langle (x + iy)^p \rangle_{R_t}, \quad c_{p,n} \in \mathbb{N}. \quad (1.9)$$

Moreover the series (1.9) is Borel summable and defines a unique function

$$M_p(t) = e^{-i\frac{\theta}{4}(1+\frac{p}{2})} t^{-1} \int_0^\infty ds \exp\left(-\frac{s}{te^{i\theta/4}}\right) b_p(s), \quad 0 \leq \theta < \pi, \quad (1.10)$$

where $b_p(s)$ is the Borel sum of (1.9) for $\alpha > 0$. Third we show

$$M_p(t) = \langle x^p \rangle_{\rho_t}, \quad \text{for all } t \geq 0. \quad (1.11)$$

In other words the Borel resummation of the short time asymptotic expansion (1.9) correctly captures the dynamics of the complex measure (1.5) aimed at, including its equilibrium aspects.

Based on (1.9) one might expect that the same holds true for the real measure defined by (1.6). However, by direct numerical simulation we find

$$\begin{aligned} M_p(t) &= \langle (x + iy)^p \rangle_{R_t}, \quad 0 \leq t \leq t_c(A_I), \\ M_p(t) &\neq \langle (x + iy)^p \rangle_{R_t}, \quad t > t_c(A_I), \end{aligned} \quad (1.12)$$

where the ‘breakdown’ time $t_c(A_I)$ depends on the strength A_I of the imaginary noise. Generically therefore the conjectured equality (1.8) holds for a finite time interval only, rendering its use to define the left hand side of (1.1) in terms of the right hand side problematic. The breakdown time increases powerlike as $A_I \rightarrow 0$. Taking A_I strictly zero may lead to an increased sensitivity on initial conditions in the Langevin simulations [6, 9]. A refined version of the conjecture (1.8) thus has to read

$$\langle \mathcal{O} \rangle_{\rho_t} \stackrel{?}{=} \lim_{A_I \rightarrow 0} \langle \mathcal{O} \rangle_{R_t}, \quad 0 \leq t \leq \lim_{A_I \rightarrow 0} t_c(A_I). \quad (1.13)$$

For practical purposes (1.13) may suffice provided the temporal variations become small before $\lim_{A_I \rightarrow 0} t_c(A_I)$ is reached [12]. A theoretical foundation of the method however requires a proof that $\lim_{A_I \rightarrow 0} t_c(A_I) = \infty$.

The article is organized as follows. In Section 2 we show that the moments $\langle x^p \rangle_{\rho_t}$ admit a transfer operator representation in terms of a non-selfadjoint propagation kernel $(e^{-t\mathbf{H}})(q, q')$ whose properties we examine in detail. In particular $(e^{-t\mathbf{H}})(q, q')$ is shown to admit a well-defined spectral representation whose norm-convergence is governed by Davies’ spectral norms [19, 20, 21]. It follows that the kernel’s $t \rightarrow \infty$ limit is well-defined and correctly projects onto its ground state proportional to $e^{-S(q)/2}$. In Section 3 we derive the results (1.9) – (1.11) for the short time asymptotics and its Borel resummation. Section 4 is devoted to the numerical simulation of the moments leading to (1.12) and the refined conjecture (1.13). For $A_I > 0$ the numerical moments also have finite large t limits which however differ from the correct answer $\lim_{t \rightarrow \infty} \langle x^p \rangle_{\rho_t}$. To ascertain the disagreement we study in Section 5 directly the spectrum and the ground state of the real Fokker-Planck operator \mathbb{P} . Both are seen to be compatible with the working hypothesis that $t \mapsto e^{t\mathbb{P}}$ indeed defines a strongly continuous semigroup with a pointwise non-negative kernel for all $A_I > 0$. Its ground state φ_0 is then used to independently compute the asymptotic values $\lim_{t \rightarrow \infty} \langle (x + iy)^p \rangle_{R_t}$ as $\langle (x + iy)^p \rangle_{\varphi_0}$. Their agreement leaves a non-naive action of the semi-group $e^{t\mathbb{P}^T}$ on holomorphic functions as the likely culprit for the failure of (1.8) for $A_I > 0$. Two appendices contain supplementary material: in Appendix A the noninteracting case is discussed and the complex Langevin method is shown to work perfectly. The under-determination of the observable flow mentioned above gives rise to an interesting parallelism between the Parisi-Klauder conjecture and quantum mechanical ‘supertasks’ [25] which we describe in Appendix B.

2. The transfer operator for the complex sextic oscillator

The averages based on the complex measure admit a transfer operator realization akin to a quantum mechanical system in the stochastic time t :

$$\langle \mathcal{O} \rangle_{\rho_t} = \frac{\int dq dq' \mathcal{O}(q) e^{S(q)/2} (e^{-t\mathbf{H}})(q, q') e^{-S(q')/2} \rho_0(q')}{\int dq dq' e^{S(q)/2} (e^{-t\mathbf{H}})(q, q') e^{-S(q')/2} \rho_0(q')}. \quad (2.1)$$

The operators $e^{-t\mathbf{H}}$, $t > 0$, generate a semigroup with integral kernel $(e^{-t\mathbf{H}})(q, q')$ to which we will refer to as the ‘complex propagation kernel’. Since \mathbf{H} is not selfadjoint for $\theta \neq 0$ none of the usual properties of a transfer operator semigroup can be taken for granted. In Section 2.1 we investigate the spectrum of \mathbf{H} and in Section 2.2 the spectral norms governing its norm-convergence properties for $t \rightarrow \infty$.

We begin by defining \mathbf{H} . The complex Fokker-Planck operator \mathbf{P} in (1.4) is not symmetric even when specialized to a real action. By a similarity transformation it can be mapped into a conventional Schrödinger type operator which is selfadjoint for real actions. Starting from \mathbf{P} in (1.4) we define

$$\mathbf{H} := -e^{S(q)/2} \mathbf{P} e^{-S(q)/2}, \quad \mathbf{P} e^{-S(q)} = 0, \quad \mathbf{H} e^{-S(q)/2} = 0. \quad (2.2)$$

This gives $\mathbf{H} = p^2 + V_{\text{FP}}(q)$ with the ‘Fokker-Planck’ potential

$$V_{\text{FP}} = \frac{1}{4} \left(\frac{\partial S}{\partial q} \right)^2 - \frac{1}{2} \frac{\partial^2 S}{\partial q^2}. \quad (2.3)$$

The associated Schrödinger operator factorizes $\mathbf{H} = (-\partial_q + \frac{1}{2}\partial_q S)(\partial_q + \frac{1}{2}\partial_q S)$ and has $\exp\{-\frac{1}{2}S(q)\}$ as exact ground state with zero energy; a feature related to an underlying supersymmetry, see [4, 3] for reviews.

For selfadjoint Schrödinger operators $\mathbf{H} = p^2 + V(q)$, with $V(q)$ a real even polynomial of degree $2p$, $p \geq 1$, the spectrum is known to be positive, purely discrete, and nondegenerate. A complete set of real-valued orthonormal eigenfunctions ψ_n , $n \geq 0$, exists and the transfer operator (Euclidean signature propagation kernel) has the spectral decomposition

$$(e^{-\frac{t}{2}\mathbf{H}})(q, q') = \sum_{n \geq 0} e^{-\frac{t}{2}E_n} P_n(q, q'). \quad (2.4)$$

The E_n are the eigenvalues of \mathbf{H} and $P_n(q, q) = \psi_n(q)\psi_n(q')$ is the projector onto the n -th eigenspace. In particular the semigroup $t \mapsto \exp\{-t\mathbf{H}\}$, $t > 0$, is strongly continuous and the $t \rightarrow \infty$ limit converges strongly to the projector P_0 onto the ground state. When the couplings parameterizing the potential V become complex, the operator \mathbf{H} is no longer

selfadjoint and none of the above properties can be taken for granted. In fact, in addition to the spectrum becoming complex, several other new phenomena occur for complex couplings, originally explored by E.B. Davies in the harmonic case [20].

In the following we consider the family of hamiltonians

$$\mathbf{H} = p^2 - e^{i\frac{\theta}{2}}\omega^2 q^2 + \lambda e^{i\theta} q^6, \quad 0 \leq \omega, \quad 0 < \lambda, \quad 0 \leq \theta < \pi, \quad (2.5)$$

with $p = -i\partial/\partial q$. The parameterization is chosen such that the complex quartic action

$$S = \frac{1}{2}\sqrt{\lambda}e^{i\theta/2}q^4, \quad (2.6)$$

gives rise to (2.5) with $\omega^2 = 3\sqrt{\lambda}$, and θ is the Wick rotation angle of the original problem. We often keep ω^2 as an independent coupling as many structural properties continue to hold for generic ω^2 . In addition to $\omega^2/\sqrt{\lambda} = 3$ a sequence of other coupling ratios, namely $\omega^2/\sqrt{\lambda} = 4k + 2\nu + 1$, $k \geq 0$, $\nu = \pm 1$, gives rise to Schrödinger equations which are ‘quasi-integrable’ in the sense that the first k solutions of a given parity ν have the form $\psi(q) = P_{k-1}(q) \exp\{-\sqrt{\lambda}q^4/4\}$, with P_{k-1} a polynomial of degree $k-1$. Responsible for this phenomenon is an underlying dynamical sl_2 symmetry, see [16] and the references therein. We aim at understanding the structure of the spectrum, of the spectral projections, and of the transfer operator associated with the hamiltonians (2.5) as a function of θ . We begin with the spectrum.

2.1 Spectrum

As mentioned, for $\theta = 0$ the spectrum of \mathbf{H} is positive, purely discrete, and non-degenerate. For complex couplings the structure of the discrete spectrum can be understood from a scaling argument. Rewriting the eigenvalue equation for $\theta = 0$, i.e. $\mathbf{H}|_{\theta=0}\psi = E\psi$, $\omega \geq 0$, in terms of $z = \exp\{-i\theta/8\}q$ one finds $\mathbf{H}\Omega = \exp\{i\theta/4\}E\Omega$, with $\Omega(z) := \psi(\exp\{i\theta/8\}z)$. Assuming that the angle θ is restricted such that normalizability is preserved this determines the θ -dependence of the eigenvalues. Writing $E_n(\omega^2, \lambda, \theta)$, $n \geq 0$, for the eigenvalues of \mathbf{H} the relation

$$E_n(\omega^2, \lambda, \theta) = e^{i\frac{\theta}{4}}E_n(\omega^2, \lambda, 0), \quad \omega^2 \geq 0, \quad \lambda > 0, \quad (2.7)$$

links the discrete spectra for real and complex couplings. Normalizability is preserved for $0 \leq \theta < \pi$, as anticipated in (2.5): by substitution into the differential equation one sees that normalizable wave functions of a λq^p potential have a dominant $\exp\{-2\sqrt{\lambda}q^{1+p/2}/(p+2)\}$ decay. For $p = 6$ and $q = \exp\{i\theta/8\}z$ the exponential is damping in z as long as $e^{i\theta/2}z^4$ has a positive real part. Incidentally (2.7) also provides a means to define the discrete spectrum of the ‘wrong sign’ sextic anharmonic oscillator as $\lim_{\theta \rightarrow \pi^-} E_n(0, \lambda, \theta) =$

$\frac{1}{\sqrt{2}}(1+i)E_n(0, \lambda, 0)$. This leads to a discrete spectrum located on the diagonals of the right half plane, in contrast to the $E_n(0, \lambda, \pi) = E_n(0, -\lambda, 0)$ which are real but unbounded from below.

To the best of our knowledge the energy levels $E_n(3\sqrt{\lambda}, \lambda, 0)$ have not been computed before. The factorization $\mathbf{H}|_{\omega=\sqrt{3\lambda}} = (-\partial_q + \sqrt{\lambda}q^3)(\partial_q + \sqrt{\lambda}q^3)$ shows that

$$\Omega_0(q) = \frac{2^{3/8}}{\Gamma(1/4)^{1/2}} \lambda^{1/16} \exp \left\{ -\frac{\sqrt{\lambda}}{4} q^4 \right\}, \quad (2.8)$$

is the unique normalized ground state with energy $E_0 = 0$. To get the excited state energies we use a simple but reliable technique: for a suitable basis on L^2 the matrix elements of \mathbf{H} define a matrix operator. Diagonalizing truncations of this infinite dimensional matrix produces approximate eigenvalues whose accuracy can be tested by probing for truncation independence. A natural choice of basis are the Hermite functions. For $\lambda > 0$ they no longer capture the qualitative behavior of the exact eigenfunctions but the resulting matrices have a band structure with only a few diagonals populated which is numerically advantageous.

To compute the corresponding matrix elements we express the hamiltonians and the Hermite functions in terms of creation and annihilation operators. With the normalizations $q = (a^* + a)/\sqrt{2\omega}$, $p = i(a^* - a)\sqrt{\omega/2}$, $\omega > 0$, $[a, a^*] = 1$, the Hermite functions are given by $|n\rangle = \frac{1}{\sqrt{n!}} a^{*n} |0\rangle$, $a|0\rangle = 0$. Converting (2.5) into a normal ordered expression in terms of a^*, a the matrix elements between the hermite states are readily obtained and read

$$\begin{aligned} \langle m | \mathbf{H} | n \rangle &= \frac{\omega}{2} (1 - e^{\theta/2}) (2n+1) \delta_{m,n} - \frac{\omega}{2} (1 + e^{\theta/2}) [\sqrt{n(n-1)} \delta_{n-2,m} + \sqrt{m(m-1)} \delta_{m-2,n}] \\ &+ \frac{\lambda}{8\omega^3} \sqrt{\frac{m!}{n!}} \left\{ \delta_{m-6,n} + (6m-9) \delta_{m-4,n} + 15(m^2-m+1) \delta_{m-2,n} \right\} + (m \leftrightarrow n) \\ &+ \frac{\lambda}{8\omega^3} (20m^3 + 30m^2 + 40m + 15) \delta_{m,n}. \end{aligned} \quad (2.9)$$

The low lying parts of the spectrum can now be computed by directly diagonalizing the hamiltonian matrices truncated to $0 \leq m, n \leq N$. Stability of the spectrum with increasing N indicates the reliability of the approximative result.

As a test we recomputed the ground state energy of \mathbf{H} with $\omega = 1$ for $10^{-3} \leq \lambda \leq 10^3$, where high accuracy results are available in the literature, see e.g. [15]. For all but very large λ truncations of $N = 150$ are sufficient to obtain the eigenvalues to 6 digits accuracy. The results are in perfect agreement with those tabulated in [15]. Note that the widely used Hill determinant method occasionally fails for sextic potentials [16].

In Table 1 we present results for the low lying eigenvalues in the ‘Fokker-Planck’ case $\omega^2 = 3\sqrt{\lambda}$. Both the θ -dependence and the λ dependence can be extracted analytically, see (2.7) and below. The overall structure of the spectrum comes out as

$$E_n = C_n e^{i\theta/4} \lambda^{1/4}, \quad C_n \rightarrow C^6 n^{3/2}, \quad \lambda > 0, \quad 0 \leq \theta < \pi, \quad (2.10)$$

so that it suffices to know the C_n . One has $C_0 = 0$ and for $n \leq 20$ a 9 digit accuracy can be achieved with truncations $N \leq 500$. We only present the first ten to six digits. The limiting behavior as $n \rightarrow \infty$ follows from a semiclassical analysis, the constant C is known analytically, see (2.22).

n	1	2	3	4	5
C_n	1.935 482	6.298 496	11.680 971	18.042 635	25.254 605
n	6	7	8	9	10
C_n	33.226 111	41.891 010	51.197 908	61.105 360	71.579 037

Table 1: C_n for the eigenvalues E_n of \mathbf{H} with $\omega^2 = 3\sqrt{\lambda}$. The truncation size is $150 \leq N \leq 300$, the last digits are rounded.

2.2 The complex propagation kernel

From the derivation of the phase relation (2.7) one sees that the eigenfunctions Ω_n of (2.5) are related to those, ψ_n , of the selfadjoint $\mathbf{H}|_{\theta=0}$ by $\Omega_n(q) = \psi_n(e^{i\theta/8}q)$. As a consequence the Ω_n are no longer orthonormal with respect to the L^2 inner product $\langle \psi, \varphi \rangle = \int dq \psi(q)^* \varphi(q)$. Rather the set Ω_n^*, Ω_n , $n \geq 0$, forms a bi-orthogonal basis in L^2 [19],

$$\langle \Omega_n^*, \Omega_m \rangle = \int dq \Omega_n(q) \Omega_m(q) = \delta_{m,n}. \quad (2.11)$$

The quantities

$$\langle \Omega_n, \Omega_n \rangle = \int dq \Omega_n(q)^* \Omega_n(q) =: N_n(\theta), \quad (2.12)$$

can be interpreted as the norms of projectors $P_n \psi = \Omega_n \langle \Omega_n^*, \psi \rangle$, which satisfy

$$\begin{aligned} P_n P_m &= \delta_{n,m} P_n, \quad \langle P_n \psi, P_m \psi \rangle = \langle \psi, \Omega_n^* \rangle \langle \Omega_n, \Omega_m \rangle \langle \Omega_m^*, \psi \rangle, \\ 1 \leq \|P_n\| &:= \sup_{\psi} \frac{\langle P_n \psi, P_n \psi \rangle^{1/2}}{\langle \psi, \psi \rangle^{1/2}} = N_n. \end{aligned} \quad (2.13)$$

The inequality follows by specializing to $\psi = \Omega_n$, the Cauchy-Schwarz inequality gives $\|P_n\| \leq N_n$, and specialization to $\psi = \Omega_n^*$ enforces equality. The *spectral norms* (2.12) originally introduced by E.B. Davies for the complex harmonic oscillator encode information about the quasi-spectrum and the norm convergence of the heat semigroup generated by the non-selfadjoint hamiltonian under consideration. For the complex harmonic oscillator it was shown in [20] that $\lim_{n \rightarrow \infty} \frac{1}{n} \ln N_n = \gamma(\theta) < \infty$, with an explicitly known constant $\gamma(\theta)$. In Appendix A we present a simple generating formula for the N_n of the complex harmonic oscillator from which the before-mentioned asymptotics can also be understood. For anharmonic oscillators it is only known that the N_n grow super-polynomially [21].

Our goal in the following is to determine the rate of growth of the N_n for the Fokker-Planck hamiltonian \mathbf{H} with $\omega^2 = 3\sqrt{\lambda}$. In a first step we show that for the class of hamiltonians (2.5) the N_n only depend on $\omega^2/\sqrt{\lambda}$ and θ ,

$$N_n(\omega^2, \lambda, \theta) = N_n\left(\frac{\omega^2}{\sqrt{\lambda}}, 1, \theta\right). \quad (2.14)$$

To this end we consider the scaling isometry

$$(S_\lambda \psi)(q) = \lambda^{1/16} \psi(\lambda^{1/8} q), \quad \langle S_\lambda \varphi, S_\lambda \psi \rangle = \langle \varphi, \psi \rangle, \quad (2.15)$$

and note that $S_{\lambda^{-1}} q S_\lambda = \lambda^{-1/8} q$. Hence

$$S_{\lambda^{-1}} \mathbf{H} S_\lambda = \lambda^{1/4} \mathbf{H} \Big|_{\omega^2 \rightarrow \omega^2/\sqrt{\lambda}, \lambda \rightarrow 1}, \quad \lambda > 0. \quad (2.16)$$

Rewriting $\mathbf{H} \Omega_n = E_n \Omega_n$ as $S_{\lambda^{-1}} \mathbf{H} S_\lambda (S_{\lambda^{-1}} \Omega_n) = E_n (S_{\lambda^{-1}} \Omega_n)$ and combining (2.16) with the fact that the spectrum is nondegenerate gives

$$E_n(\omega^2, \lambda) = \lambda^{1/4} E_n\left(\frac{\omega^2}{\sqrt{\lambda}}, 1\right), \quad S_{\lambda^{-1}} \Omega_n = \Omega_n \Big|_{\omega^2 \rightarrow \omega^2/\sqrt{\lambda}, \lambda \rightarrow 1}. \quad (2.17)$$

On the other hand S_λ is an isometry, so that

$$\begin{aligned} \langle \Omega_n, \Omega_n \rangle &= N_n = \langle \Omega_n, \Omega_n \rangle \Big|_{\omega^2 \rightarrow \omega^2/\sqrt{\lambda}, \lambda \rightarrow 1}, \\ \langle \Omega_n^*, \Omega_n \rangle &= 1 = \langle \Omega_n^*, \Omega_n \rangle \Big|_{\omega^2 \rightarrow \omega^2/\sqrt{\lambda}, \lambda \rightarrow 1}, \end{aligned} \quad (2.18)$$

which establishes (2.14).

Next we observe that the pointwise defined spectral sum

$$(e^{-\frac{t}{2}\mathbf{H}})(q, q') := \sum_{n \geq 0} e^{-\frac{t}{2}E_n} P_n(q, q'), \quad (2.19)$$

with E_n the eigenvalues in (2.7), is a candidate for the kernel of the transfer operator. The N_n 's in principle then are the coefficients in the expansion of the complex partition function

$$\text{Tr}[e^{-\frac{t}{2}\mathbf{H}}] \stackrel{?}{=} \sum_{n \geq 0} e^{-\frac{t}{2}E_n} N_n. \quad (2.20)$$

Since the 'tail' of the partition function corresponding to large quantum numbers n should be dominated by semiclassical configurations one expects that the rate of growth of the N_n 's can be extracted from a WKB-type evaluation of the partition function.

We begin by reconsidering the eigenvalues. For Schrödinger operators $p^2 + V(q)$, with $V(q)$ a real even polynomial of degree $2p$, $p \geq 1$, the spectrum is known to be purely discrete and nondegenerate and to scale like $E_n \sim C n^{2p/(p+1)} + O(n^{(p-1)/(p+1)})$, for large quantum numbers n , see e.g. [17]. Combined with the scaling law (2.7) one obtains for the eigenvalues E_n of the hamiltonians (2.5) a scaling behavior

$$E_n \sim C n^{3/2} + O(n^{1/2}). \quad (2.21)$$

In the Fokker-Planck case, $\omega^2 = 3\sqrt{\lambda}$, it follows from (2.17) that the exact eigenvalues have the form anticipated in (2.10), where only the constants C_n remain to be determined. One expects their large n behavior to be governed by a suitable semi-classical approximation. Indeed, application of the SUSY WKB formula [23] gives for the semi-classical eigenvalues $\epsilon_n(\lambda)$ the simple expression

$$\epsilon_n(\lambda) = C^6 n^{3/2} \lambda^{1/4}, \quad C = \left(\frac{\sqrt{\pi} \Gamma(5/3)}{\Gamma(7/6)} \right)^{1/4} \approx 1.14599. \quad (2.22)$$

The λ -dependence is evidently of the form mandated by the scaling law (2.17) while the n -dependence is in accordance with (2.21). Comparing with (2.10) one sees that SUSY WKB yields an expression for the limiting constant $\lim_{n \rightarrow \infty} n^{-3/2} C_n$. In fact, the C_n approach their asymptotic values fairly quickly, rendering (2.22) a good approximation to the spectrum. Computing the C_n 's as in Table 1 the ratio $C_n/(n^{3/2} C^6)$ comes out as: 0.854 484, 0.999 311, 0.999 993, at $n = 1, 10, 100$, respectively. The standard WKB formula, in contrast, gives the correct $n \rightarrow \infty$ asymptotics, but a far worse description for small n .

Motivated by the good description of the spectrum by the SUSY WKB approximation, we also consider the associated wave functions. Evaluating the result of [24] in the case at hand one finds that the quasi-classical eigenfunctions depend for real λ on q only through the combination

$$z = \lambda^{1/8} n^{-1/4} C^{-1} q, \quad (2.23)$$

such that the domain of definition in q corresponds to $z \in [-1, 1]$ in z . The original norm as defined by a q -integral with (n, λ) -dependent domain translates into

$$N_n(0) := \frac{2}{\tau_0 C^2} \int_{-1}^1 dz \Omega_n(z) \Omega_n(z), \quad \tau_0 = \frac{2\pi^{1/4} \Gamma[7/6]^{3/2}}{\Gamma[2/3] \Gamma[5/6]^{1/2}} \approx 1.84928. \quad (2.24)$$

The normalizations are such that $N_n(0) \rightarrow 1$ for $n \rightarrow \infty$. However the inner products with $n \neq m$ do *not* approach zero as $n + m \rightarrow \infty$ with fixed $n - m$. The quasi-classical eigenfunctions for real λ therefore do not form an orthonormal set. The explicit expression for the n -th quasi-classical eigenfunction comes out as

$$\begin{aligned} \Omega_n(z) &= \frac{n^{-1/4} \lambda^{1/16}}{(1 - z^6)^{1/4}} \cos \left[\frac{\pi}{2} n + n z f(z) + \frac{1}{2} \arcsin z^3 \right], \\ f(z) &:= C^4 {}_2F_1 \left[-\frac{1}{2}, \frac{1}{6}, \frac{7}{6}; z^6 \right] = -0.047\,460\,z^6 - 0.002\,820\,z^{12} + O(z^{18}). \end{aligned} \quad (2.25)$$

Note that $(S_{\lambda^{-1}}\Omega_n)(z)$ is λ -independent, as required by (the quasi-classical counterpart of) (2.17). One can now readily restore the θ -dependence and evaluate (2.24) with integrand $|\Omega_n(z)|^2$, which defines $N_n(\theta)$ in the quasi-classical approximation. Straightforward numerical integration then shows convincingly

$$\frac{1}{n} \ln N_n(\theta) \rightarrow \gamma(\theta) < \infty. \quad (2.26)$$

As an additional test of (2.26) we also investigate the scaling of $\ln N_n$ by direct numerical evaluation using an extension of the truncation technique employed for the eigenvalues. On account of (2.14) it suffices to evaluate the N_n 's and their asymptotics for one λ . Since the interplay between the different normalizations is crucial, we spell out the details here: inserting a resolution of the identity in terms of real Hermite functions $|k\rangle$, $k \in \mathbb{N}_0$, into the eigenvalue equation for \mathbf{H} , i.e.

$$\sum_j \langle k | \mathbf{H} | j \rangle \langle j | \Omega_n \rangle = E_n \langle k | \Omega_n \rangle, \quad (2.27)$$

one identifies the numerically computed eigenvectors of the truncated hamiltonian matrix as

$$v_n^{(j)} = c_n \langle j | \Omega_n \rangle, \quad j = 1, \dots, N, \quad n \in \mathbb{N}_0, \quad c_n \in \mathbb{C}. \quad (2.28)$$

Assuming that the exact eigenfunctions Ω_n form a bi-orthogonal basis normalized according to (2.11), (2.12) one expects

$$\sum_j |v_n^{(j)}|^2 \rightarrow |c_n|^2 N_n, \quad \sum_j [v_n^{(j)}]^2 \rightarrow c_n^2, \quad N \rightarrow \infty. \quad (2.29)$$

Numerical diagonalization routines typically produce eigenvectors normalized to have unit norm in \mathbb{C}^N . Based on (2.29) the N_n can then be obtained via

$$\left| \sum_j [v_n^{(j)}]^2 \right| \rightarrow \frac{1}{N_n}, \quad N \rightarrow \infty. \quad (2.30)$$

We again consider the Fokker-Planck case $\omega^2 = 3\sqrt{\lambda}$ in detail. From (2.8) one computes

$$N_0 = \frac{1}{(\cos \frac{\theta}{2})^{1/4}}, \quad \text{for all } \lambda > 0. \quad (2.31)$$

The $N_n, n \leq n_0$, we compute numerically via (2.30). In a first step we verify the λ -independence for $N = 500$. Comparing the results for $\lambda = 10^{-3}, 10^{-1}, 1, 10, 10^3$, one finds

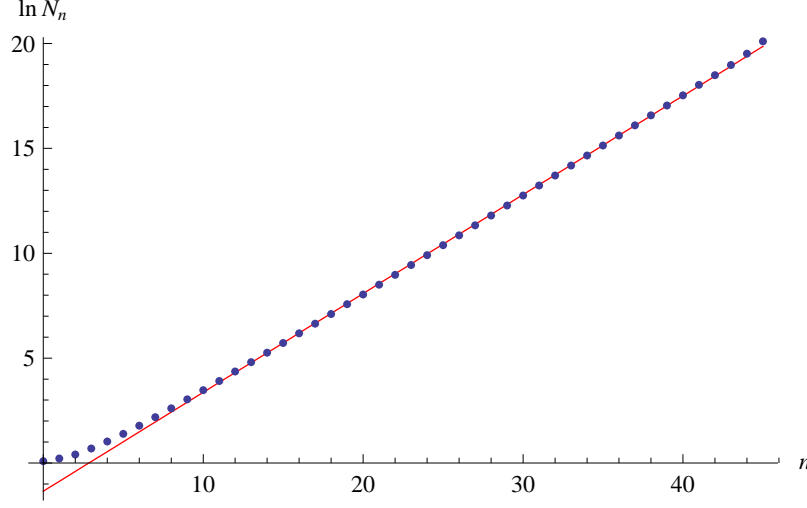


Figure 1: $\ln N_n$ versus linear fit, $-1.34 + 0.47 n$.

that whenever the N_n 's are numerically stable they are also λ -independent to the same accuracy. Specifically, for θ less than $\pi/4$ at least N_n , $n = 0, \dots, 30$, are reliable and λ -independent at the 10^{-5} level, while for larger θ only N_n , $n = 0, \dots, 10$, or so are reliable and λ -independent at the same accuracy level. Next we fix $\lambda = 1$ and evaluate the N_n 's from $N = 1000, 1300, 1500$ truncations. The results for $\theta = \pi/2$ are reported in Table 2.

n	0	1	2	3	4	5	6	7
$\log N_n$	0.08664	0.21303	0.40745	0.69222	1.02414	1.3901	1.7800	2.1869
n	8	9	10	15	20	25	35	45
$\ln N_n$	2.6063	3.0351	3.4711	5.7206	8.0350	10.3840	15.139(1)	20.105(3)

Table 2: Logarithm of spectral norms at $\theta = \pi/2$.

As visible from Fig. 1, a scaling of $\ln N_n$ linear in n is favored already for moderately large n , consistent with the previous result (2.26).

The scaling law (2.26) has implications for the structure of the resolvent and the quasi-spectra of \mathbf{H} . In our context (2.26) implies that the spectral representation (2.19) is norm convergent only if $t > t_*$ and divergent otherwise, where using (2.10)

$$t_* = \frac{2}{\lambda^{1/4} \cos \frac{\theta}{4}} \max_n \frac{\ln N_n}{C_n} < \infty. \quad (2.32)$$

Importantly the $t \rightarrow \infty$ limit still projects onto the ground state for *all* $0 \leq \theta < \pi$:

$$s - \lim_{t \rightarrow \infty} e^{-\frac{t}{2}(\mathbf{H} - E_0)} = P_0, \quad \lim_{t \rightarrow \infty} e^{-\frac{t}{2}(\mathbf{H} - E_0)} \psi = \Omega_0 \langle \Omega_0^*, \psi \rangle, \quad \text{a.e.} \quad (2.33)$$

This covers the range needed for the Wick rotation and allows one to define the Lorentzian signature propagation kernel in terms of $\theta \rightarrow \pi_-$. The result is somewhat surprising as for $\pi/2 < \theta < \pi$ the potential in (2.5) becomes unbounded from below. In (2.33) the first equation implies the second by Cauchy-Schwarz and the strict positivity of $|\Omega_0(q)|$. In extension of the results for the complex harmonic oscillator [20, 21, 22] one expects (2.26) also to govern the behavior of $(t, \theta) \mapsto e^{-t\mathbf{H}}$ as a bounded holomorphic semigroup.

Finally, the analysis leads to an important scaling relation for the averages (2.1). It suffices to consider the monomials $\mathcal{O}(q) = q^p$ with p even. Writing $m_p(t) = \langle q^p \rangle_{\rho_t}$ and assuming that the initial value distribution $\rho_0(q)$ is for complex coupling replaced with $\rho_0(e^{i\theta/8}q)$ one finds

$$m_p(t) = e^{-ip\theta/8} m_p|_{\theta=0}(e^{i\theta/4}t), \quad (2.34)$$

by combining (2.19), (2.16) and (2.7).

3. Borel resummation of the short time asymptotics

We now resume the investigation of the putative identity

$$\langle \mathcal{O} \rangle_{\rho_t} \stackrel{?}{=} \langle \mathcal{O} \rangle_{R_t}, \quad t \geq 0, \quad (3.1)$$

as surveyed in the introduction. The results of Section 2 entail that the left hand side of (3.1) is well-defined for all $t \geq 0$ and converges to the desired $t \rightarrow \infty$ average even if the underlying quartic action is complex. Our strategy for investigating (3.1) is based on the fact that both sides have the same $t \rightarrow 0$ asymptotic expansion, as detailed in Section 3.2. For the quartic selfinteraction the coefficients of the expansion turn out to be such that the series is amenable to a Borel resummation. The Borel transform defines a unique function for all $0 \leq \theta < \pi$, which is shown to coincide with the left hand side of (3.1) in Section 3.3. Nevertheless this falls short of proving (3.1). In order to disentangle the issues involved we recap briefly the heuristic arguments for the validity of the conjecture.

3.1 Recap of the Parisi-Klauder conjecture

The rationale for the definition of \mathbb{P} in (1.6) can be understood by rewriting it in complex coordinates, $z = x + iy$, $\bar{z} = x - iy$. This gives

$$\mathbb{P} = [\partial_z^2 + \partial_z S \partial_z + \partial_z (\partial_z S)] + [\partial_{\bar{z}}^2 + (\partial_z S)^* \partial_{\bar{z}} + \partial_{\bar{z}} (\partial_z S)^*] + 2(A_R + A_I) \partial_z \partial_{\bar{z}}. \quad (3.2)$$

We may assume $S(z)^* = S^*(\bar{z})$, where S^* has complex conjugated coefficients, e.g. $S^*(z) = \sum_{n \geq 0} a_n^* z^n$ for $S(z) = \sum_{n \geq 0} a_n z^n$. One sees the form of \mathbb{P} is dictated largely by the

requirement that it acts like \mathbf{P} on holomorphic functions and maps real functions to real functions. The mixing term can formally be removed by taking $A_R + A_I = 0$, i.e. $A_R = 1/2$, $A_I = -1/2$. This however compromises the standard real decomposition of the complex Langevin equation and even in the free case leads to equilibrium distributions which are not integrable, see Appendix A. In principle the mixed term could be replaced with $\nu(z\bar{z})\partial_z\partial_{\bar{z}}$ for any function ν of one variable without affecting the reality of the operator or its action on holomorphic functions.

The standard heuristic argument for the validity of (3.1) proceeds by contour deformation:

$$\frac{\rho_t(x)}{\int dx \rho_t(x)} = \frac{\int dy R_t(x - iy, y)}{\int dx dy R_t(x, y)}, \quad (3.3)$$

relates both averages provided $z \mapsto R_t(z - iy, y)$ is for fixed $y \in \mathbb{R}$ analytic with suitable fall-off:

$$\begin{aligned} \int_{\mathbb{R}^2} dx dy \mathcal{O}(x + iy) R_t(x, y) &= \int_{\mathbb{R}} dy \int_{\mathbb{R} + iy} dz \mathcal{O}(z) R_t(z - iy, y) \\ &= \int_{\mathbb{R}} dx \mathcal{O}(x) \int_{\mathbb{R}} dy R_t(x - iy, y). \end{aligned} \quad (3.4)$$

In particular, if $R_t(x, y)$ itself has a $t \rightarrow \infty$ limit $\varphi_0(x, y)$, it should obey

$$\frac{e^{-S(x)}}{\int dx e^{-S(x)}} = \frac{\int dy \varphi_0(x - iy, y)}{\int dx dy \varphi_0(x, y)}. \quad (3.5)$$

For convenience we note the resulting asymptotic form of (3.1) explicitly: there exists an integrable nonnegative solution $\varphi_0(x, y)$ of $\mathbb{P}\varphi_0 = 0$ such that for \mathcal{O} of polynomial growth

$$\langle \mathcal{O}(x) \rangle_{e^{-S}} \stackrel{?}{=} \langle \mathcal{O}(z) \rangle_{\varphi_0}, \quad (3.6)$$

holds, where

$$\langle \mathcal{O}(x) \rangle_{e^{-S}} := \frac{\int dx \mathcal{O}(x) e^{-S(x)}}{\int dx e^{-S(x)}}, \quad \langle \mathcal{O}(z) \rangle_{\varphi_0} := \frac{\int dx dy \mathcal{O}(x + iy) \varphi_0(x, y)}{\int dx \varphi_0(x, y)}. \quad (3.7)$$

Under the assumption that $R_t(x, y)$ has a spectral resolution of the form $R_t(x, y) = \sum_n e^{-tE_n} c_n \varphi_n(x, y)$, $\mathbb{P}\varphi_n = E_n \varphi_n$, the large t limit in (1.7) will be dominated by the $n = 0$ term and $e^{-tE_0} c_0$ will drop out in the ratio (1.7). The mere existence of a limit $\lim_{t \rightarrow \infty} \langle \mathcal{O}(z) \rangle_{R_t} = \langle \mathcal{O}(z) \rangle_{\varphi_0}$ therefore only requires $\text{Re} E_n \geq 0$, for all n , not necessarily $E_0 = 0$. Likewise the heuristic contour shift (3.4) and its limiting version (3.5) does not require $E_0 = 0$. On general grounds $E_0 = 0$ must lie in the spectrum of \mathbb{P} , see (3.13) below,

and $E_0 = 0$ is also the only ground state energy of \mathbb{P} compatible with (3.6). In contrast to \mathbf{P} in (1.4) where e^{-S} manifestly is a zero mode, the solutions of $\mathbb{P}\varphi_0 = E_0\varphi_0$ with φ_0 integrable and non-negative can in general not be found analytically, for any candidate E_0 , so the mere existence of an appropriate φ_0 in the kernel of \mathbb{P} is nontrivial.

The putative ground state wave function φ_0 has to obey an additional consistency condition. Denoting by \mathbb{P}^T the real adjoint of \mathbb{P} one has

$$E_0 \int dxdy \mathcal{O}(x+iy) \varphi_0(x,y) = \int dxdy (\mathbb{P}^T \mathcal{O})(x+iy) \varphi_0(x,y),$$

$$(\mathbb{P}^T \mathcal{O})(z) = \partial_z^2 \mathcal{O} - \partial_z S \partial_z \mathcal{O}, \quad (3.8)$$

using only $\mathbb{P}\varphi_0 = E_0\varphi_0$ and (3.2). That is: for $E_0 \neq 0$ the $\langle \cdot \rangle_{\varphi_0}$ averages of a holomorphic observable \mathcal{O} and its ‘dual’ $E_0^{-1} \mathbb{P}^T \mathcal{O}$ have to coincide. For $E_0 = 0$ the $\langle \cdot \rangle_{\varphi_0}$ averages of all observables in the image of \mathbb{P}^T have to vanish; see [12] for an alternative derivation.

In fact only the $E_0 = 0$ version of (3.8) is compatible with the validity of (3.6). To see this, take $\mathcal{O}(z) = z^2$. Then (3.8) reads $(E_0/2) \int dxdy (x+iy)^2 \varphi_0(x,y) = \int dxdy \varphi_0(x,y) - \int dxdy (z \partial_z S)(z=x+iy) \varphi_0(x,y)$, where by assumption the $\langle \cdot \rangle_{\varphi_0}$ averages can be replaced with $\langle \cdot \rangle_{e^{-S}}$ averages. This gives

$$\frac{1}{2} E_0 \langle x^2 \rangle_{e^{-S}} = 1 - \langle x \partial_x S \rangle_{e^{-S}}. \quad (3.9)$$

Upon integrations-by-parts the right hand side vanishes, enforcing $E_0 = 0$ as the only candidate ground state energy for \mathbb{P} compatible with (3.6). From a general functional analytical principle one can in fact obtain $E_0 = 0$ irrespective of the validity of (3.6), see (3.13).

Returning to (3.1) and the heuristic argument (3.4) for it, we stress that even in very simple interacting theories one has no analytic control over $R_t(x,y)$, not even for real arguments and the analyticity assumption is little more than a leap of faith. In order to highlight the nontrivial nature of the seemingly innocuous steps in (3.4) we spell out some of the mathematical underpinnings needed for $R_t(x,y)$ to be well-defined. The defining relation in (1.6) is a generalized heat equation and the operator \mathbb{P} should generate the associated semigroup. As such it must satisfy a number of necessary conditions which we first list and then comment on:

- (i) $\mathbb{P}^T \mathcal{O}(x+iy) = [(\partial_z^2 - \partial_z S \partial_z) \mathcal{O}](x+iy)$.
- (ii) \mathbb{P} generates a strongly continuous semi-group $t \mapsto e^{t\mathbb{P}} : L^1 \rightarrow L^1$, whose kernel $(e^{t\mathbb{P}})(x,y;x',y')$ is pointwise positive.
- (iii) \mathbb{P} has a unique positive ground state $\varphi_0 \in L^1$ with zero energy, $\mathbb{P}\varphi_0 = 0$.

Condition (i) is necessary for the differentiated version of (3.1) to hold, see Section 3.1. It is manifestly satisfied by the proposed real operator \mathbb{P} in (1.6), but does not uniquely

determine it. Since \mathbb{P} is not symmetric the natural functional analytical setting is that of a dual pair of Banach spaces, where \mathbb{P} acts on one space and \mathbb{P}^T on its dual. Interpreting $e^{t\mathbb{P}}$ as an operator on a weighted L^1 space a sufficient condition for boundedness is

$$\sup_{x', y'} w(x', y') \int dx dy w(x, y)^{-1} (e^{t\mathbb{P}})(x, y; x', y') < \infty, \quad (3.10)$$

for *real* x, y, x', y' and a suitable weight function $w : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ to be specified later. The initial conditions $R_0(x, y) = \rho_0(x)\delta(y)$ do not lie in L^1 but we assume that $R_t(x, y)$ is in L^1 and is smooth for all $t > 0$. The requirement that \mathbb{P} is the generator of a strongly continuous semigroup poses strong functional analytical constraints, which are however indirectly coded in the resolvent and not verifiable by inspection of the differential operator. The same holds for the even stronger condition that the evolution kernel is pointwise nonnegative. For $R_t(x, y)$ both properties in principle follow from the stochastic differential equation provided a global solution with the appropriate initial conditions exist. A necessary condition for (ii) is that \mathbb{P} has a spectrum of the form

$$\sigma(-\mathbb{P}) = \{E_r \pm iE_i, E_r, E_i \geq 0\}, \quad E_0 := \inf E_r \in \sigma(-\mathbb{P}). \quad (3.11)$$

Generally, if \mathbb{P}^T is the Banach space adjoint of \mathbb{P} with respect to a suitable pairing the full spectra of \mathbb{P} and \mathbb{P}^T in principle coincide, see e.g. [18], Thm. VI.7. An eigenvalue E_n of \mathbb{P} however can either be an eigenvalue of \mathbb{P}^T or lie in \mathbb{P}^T 's residual spectrum. Here we interpret the operators \mathbf{P}^T and \mathbb{P}^T as maps from $L^\infty \rightarrow L^\infty$, where L^∞ is dual to a weighted L^1 space. For the dual Banach spaces we take

$$L^1(\mathbb{R}^2, w^{-1} dx dy), \quad L^\infty(\mathbb{R}^2, w dx dy), \quad w(x, y) = e^{-\delta(x^2+y^2)}, \quad \delta > 0, \quad (3.12)$$

and similarly for the one variable case. The choice of the weight function ensures that all polynomials $x^p, y^p, (x + iy)^p$ and also $e^{j_1 x + j_2 y}, e^{j(x+iy)}$ for real sources are elements of the dual space. The integrability condition on the eigenfunctions of $e^{t\mathbb{P}}$ and $R_t(x, y)$ is stronger than the unweighted one, but should for polynomial actions be easily satisfied. In particular the constants are elements of the dual space and are annihilated by \mathbf{P}^T and \mathbb{P}^T , respectively. Hence $0 \in \sigma(\mathbb{P}^T)$ which explains why zero lies in the spectrum of \mathbb{P} . More generally one has the following relations among the spectra

$$0 \in \sigma(\mathbf{H}) = \sigma(-\mathbf{P}) = \sigma(-\mathbf{P}^T), \quad 0 \in \sigma(-\mathbb{P}^T) = \sigma(-\mathbb{P}), \quad (3.13)$$

but $\sigma(-\mathbf{P}^T) \subset \sigma(-\mathbb{P}^T)$ does not follow. Note that all spectra refer to different Hilbert or Banach spaces.

Finally, condition (iii) means that the ground state energy E_0 lies in the discrete spectrum of the operator. This could be violated in principle but in the present setting of scalar fields

with a polynomial interaction it is a feature one would expect to hold. For a symmetric semi-group uniqueness of a normalizable ground state follows from strict positivity of the kernel in (ii). For a non-symmetric semigroup it hinges on the validity of the $t \rightarrow \infty$ projection property. Since this will be relevant later on let us assume that the kernel in (ii) admits a spectral representation of the form

$$(e^{t\mathbb{P}})(x, y; x', y') = \sum_{n \geq 0} e^{tE_n} \varphi_n(x, y) \psi_n(x', y') \quad (3.14)$$

where $\varphi_n \in L^1$, $\psi_n \in L^\infty$, $\int dx dy \varphi_m(x, y) \psi_n(x, y) = \delta_{m,n}$ for $m, n \neq 0$, and $E_0 = 0$. For the kernel of $e^{t\mathbb{P}^T}$ then the same expansion holds with the roles of φ_n and ψ_n interchanged. The relevant projection property reads

$$R_t(x, y) = \sum_n e^{tE_n} \varphi_n(x, y) \int dx' \rho_0(x') \psi_n(x', 0) \rightarrow \varphi_0(x, y) \int dx' \rho_0(x') \psi_0(x', 0). \quad (3.15)$$

3.2 Observable flow and its small t expansion

Under the assumption that $R_t(x, y)$ has sufficient fall-off in x, y one can integrate by parts and have the transpose (real adjoint) of \mathbb{P} act on the observables. By definition of \mathbb{P} its transpose acts on holomorphic observables like $\mathbf{L}_z = \partial_z^2 - \partial_z S \partial_z$. This gives rise to a flow equation for the averages $\langle \mathcal{O} \rangle_{R_t}$. Indeed,

$$\begin{aligned} \partial_t \int dx dy \mathcal{O}(x + iy) R_t(x, y) &= \int dx dy \mathcal{O}(x + iy) \mathbb{P} R_t(x, y) \\ &= \int dx dy \mathbb{P}^T \mathcal{O}(x + iy) R_t(x, y) = \int dx dy (\mathbf{L}_z \mathcal{O})(x + iy) R_t(x, y). \end{aligned} \quad (3.16)$$

In particular the normalization factor $\int dx dy R_t(x, y)$ is time independent. The same derivation works for averages $\langle \mathcal{O} \rangle_{\rho_t}$ starting from the complex Fokker-Planck equation $\partial_t \rho_t = \mathbf{P} \rho_t$. Both averages therefore obey identical systems of ‘observable flow equations’:

$$\begin{aligned} \partial_t \langle \mathcal{O} \rangle_{R_t} &= \langle \partial_z^2 \mathcal{O} \rangle_{R_t} - \langle \partial_z S \partial_z \mathcal{O} \rangle_{R_t} . \\ \partial_t \langle \mathcal{O} \rangle_{\rho_t} &= \langle \partial_x^2 \mathcal{O} \rangle_{\rho_t} - \langle \partial_x S \partial_x \mathcal{O} \rangle_{\rho_t} . \end{aligned} \quad (3.17)$$

The initial data coincide by assumption $\langle \mathcal{O} \rangle_{R_0} = \langle \mathcal{O} \rangle_{\rho_0}$ yielding the following simple but fruitful Lemma.

Lemma: The averages $\langle \mathcal{O} \rangle_{R_t}$ and $\langle \mathcal{O} \rangle_{\rho_t}$ admit asymptotic expansions for $t \rightarrow 0$ which coincide

$$\langle \mathcal{O} \rangle_{R_t} \sim \sum_{n \geq 0} \frac{t^n}{n!} c_n \sim \langle \mathcal{O} \rangle_{\rho_t}, \quad c_n = \frac{\int dx (\mathbf{L}^n \mathcal{O})(x) \rho_0(x)}{\int dx \rho_0(x)}, \quad (3.18)$$

where $\mathbf{L} = \partial_x^2 - \partial_x S \partial_x$. Generally, to the extent the flow equations (3.17) determine the averages they must coincide.

We add some remarks: (i) The ‘observable flow equations’ generalize the Schwinger-Dyson equation for the partition function in [9, 10]. Only the equilibrium aspects (vanishing time derivatives in (3.17)) are utilized in the approach of [9, 10]. (ii) Clearly both averages coincide as formal power series in t . In the formal expansion of the R_t averages \mathbb{P}^T acts like $\mathbf{L}_z = \partial_z^2 - \partial_z S \partial_z$ on $\mathcal{O}(z)$ while in the formal expansion of the ρ_t averages \mathbf{P}^T acts like \mathbf{L} , both evidently producing the same coefficients. The c_n are also the unique solution of the recursion relations entailed by inserting a power series ansatz into (3.17). By the smoothness property of $e^{t\mathbb{P}^T}$ and $e^{t\mathbf{P}^T}$ images for $t > 0$ the exact averages do admit a series expansion in t , and its asymptotic nature readily follows from the uniqueness of the c_n . (iii) In the noninteracting case detailed in Appendix A the flow equations (3.17) completely determine the averages, implying the validity of the Parisi-Klauder conjecture without the need to explicitly compute the respective propagation kernels. (iv) In interacting situations the flow equation (3.17) will in general fall short of fully determining the flow of the averages of generic \mathcal{O} ’s. Rather upon choosing a ‘suitable complete set’ of observables the averages of a small subset will recursively determine all others. The quartic case detailed later on is a good illustration. It does not help that for both averages the time dependence can formally be attributed to the observables [12] $\partial_t \mathcal{O}(t) = (\partial_z^2 - \partial_z S \partial_z) \mathcal{O}(t)$ differing only in the interpretation as functions of z and x ; the under-determination persists. (v) A two step approach to lift the under-determination is described in [9, 10]. In a first step one characterizes solutions of the equilibrium Schwinger-Dyson equations as complexified path integrals and then tries to select the one corresponding to the complex Langevin process driven by \mathbb{P} via minimization of an effective potential. The approach does not discriminate between different values of $A_I > 0$. (vi) Our strategy focusses on situations where the series (3.21) is Borel summable, i.e. when the c_n are sign-alternating and of essentially factorial growth. Then both averages can differ only in a fairly prescribed way and the Borel transform of the series potentially coincides with the exact result for one or both of the averages.

In the framework of the observable flow the equality (1.8) comes about as follows:

$$\begin{aligned}
\int dx dy \mathcal{O}(x + iy) R_t(x, y) &= \int dx dy \mathcal{O}(x + iy) [e^{t\mathbb{P}} \rho_0(x) \delta(y)] = \\
&= \int dx dy [e^{t\mathbb{P}^T} \mathcal{O}(x + iy)] \rho_0(x) \delta(y) \stackrel{?}{=} \int dx [e^{t\mathbf{L}_x} \mathcal{O}(x)] \rho_0(x) \\
&= \int dx \mathcal{O}(x) [e^{t\mathbf{P}} \rho_0(x)] = \int dx \mathcal{O}(x) \rho_t(x).
\end{aligned} \tag{3.19}$$

In contrast to the traditional argument (3.4) the above variant does not require control over the the kernel $R_t(x, y)$ in the complex plane. The only questionable step now is the one marked with ?.

The transpositions entering the other steps should be unproblematic. For the one vari-

able semigroup this is because \mathbf{P} is similar to the real symmetric operator \mathbf{H} . One has $-e^{S/2}\mathbf{P}e^{-S/2} = \mathbf{H} = \mathbf{H}^T = -e^{-S/2}\mathbf{P}^Te^{S/2}$, with $\mathbf{P}^T = \mathbf{L}_x$, which implies for the kernels $(e^{-t\mathbf{H}})(x, x') = (e^{-t\mathbf{H}})(x', x)$ and

$$(e^{t\mathbf{P}})(x, x') = e^{S(x)/2}(e^{-t\mathbf{H}})(x, x')e^{-S(x')/2} = e^{S(x')}(e^{t\mathbf{P}})(x', x)e^{-S(x)} = e^{t\mathbf{P}^T}(x', x). \quad (3.20)$$

For the two variable case the kernel of the transposed operator is simply $(e^{t\mathbb{P}^T})(x, x'; y, y') = (e^{t\mathbb{P}})(y, y'; x, x')$, and granting the correct domains the second step in (3.21) follows. In contrast to \mathbf{P} the kernels of $e^{t\mathbb{P}}$ and $e^{t\mathbb{P}^T}$ (with the same order of arguments) can however not be related by a similarity transformation with a function (multiplication operator). Assuming otherwise and inserting an appropriate ansatz one finds $A_I\partial_x F_y = A_R\partial_y F_x$ as a necessary condition. Since $\partial_x F_y = -\partial_y F_x$ follows from the defining relations, this could hold only for $A_R + A_I = 0$. Similarity transformations with other operators are not excluded.

The step ? holds in the sense of an asymptotic series in t on account of the Lemma. For the action of the semigroups themselves, with the image supposed to be smooth functions in t , we separately highlight the corresponding property:

$$\begin{aligned} \text{(iv)} \quad & \int dx' dy' \exp(t\mathbb{P}^T)(x, 0; x', y') \mathcal{O}(x' + iy') = \int dx' \exp(t\mathbf{P}^T)(x, x') \mathcal{O}(x'), \\ & \text{for all } t > 0 \text{ and all } x. \end{aligned}$$

We regard the invalidity of this extension as the likely culprit for the failure of the method whenever it fails. The right hand side is an A_I independent function of x for all t while the kernel on the left hand side is A_I dependent. In the $t \rightarrow \infty$ limit the x dependence disappears but a mismatched normalization may have been picked up. The expected $t \rightarrow \infty$ limit of (iv) can be inferred from (3.20) and (3.15)

$$\psi_0(x, 0) \int dx' dy' \varphi_0(x', y') \mathcal{O}(x' + iy') = \langle \mathcal{O} \rangle_{e^{-S}}. \quad (3.21)$$

Consistency with (3.6) requires that $\psi_0(x, 0) = \psi_0 > 0$ is a constant given by $\psi_0^{-1} = \int dx' dy' \varphi_0(x', y')$. Only if both \mathbb{P} and \mathbb{P}^T have purely discrete spectra does the uniqueness of φ_0 imply the uniqueness of ψ_0 and since a constant trivially lies in the kernel of \mathbb{P}^T constancy in y follows as well: $\psi_0(x, y) = \psi_0$.

In Section 5 we shall construct φ_0 explicitly for the quartic selfinteraction and find that it depends nontrivially on A_I and so do its averages of holomorphic observables. Our failure diagnostics differs from the one in [12], Section 4.2, where the rapid growth of $\int dx' dy' \exp(t\mathbb{P}^T)(x, y; x', y') \mathcal{O}(x' + iy')$ in y (for the $U(1)$ link model) is argued to invalidate the integration by parts in the second equality of (3.19). We essentially define $e^{t\mathbb{P}}$ by $e^{t\mathbb{P}^T}$ acting on holomorphic observables so that validity of this step is built in. Table 5 in Section 5 also provides some direct computational evidence for its legitimacy. Nevertheless (3.21)

fails because the ground state φ_0 has the wrong structure. Thus (iv) must fail irrespective of any growth property in y or invalid integration by parts.

3.3 Temporal Borel resummation for quartic actions

For a polynomial action like $S(x) = \alpha x^4$, $\alpha = \sqrt{\lambda} e^{i\theta/2}/2$, the powers x^p , $p = 0 \bmod 2$, are a natural complete set of observables. We set

$$m_p(t) = \begin{cases} \langle z^p \rangle_{R_t} & \text{real Fokker-Planck evolution,} \\ \langle x^p \rangle_{\rho_t} & \text{complex Fokker-Planck evolution} \end{cases}. \quad (3.22)$$

The flow equations (3.17) translate into $\partial_t m_p = p(p-1)m_{p-2} - 4\alpha p m_{p+2}$, $p = 0 \bmod 2$. Since $m_0(t) = 1$, one sees that all $m_p(t)$, $p \geq 4$, are determined by $m_2(t)$ via the recursion

$$m_p = \frac{1}{4\alpha} \left[(p-3)m_{p-4} - \frac{1}{p-2} \partial_t m_{p-2} \right]. \quad (3.23)$$

The first few read

$$\begin{aligned} m_4(t) &= \frac{1}{8\alpha} [2 - \partial_t m_2], \\ m_6(t) &= \frac{1}{128\alpha^2} [\partial_t^2 m_2 + 96m_2\alpha], \\ m_8(t) &= -\frac{1}{3072\alpha^3} [\partial_t^3 m_2 + 576\alpha \partial_t^2 m_2 - 960\alpha]. \end{aligned} \quad (3.24)$$

The flow equations (3.17) therefore entail that all moments in (3.22) coincide for all t if the second moments coincide, $\langle z^2 \rangle_{R_t} = \langle x^2 \rangle_{\rho_t}$, for all t . This structure gives rise to an amusing parallelism to the quantum mechanical supertasks introduced by Norton [25] which we outline in Appendix B.

Either by solving (3.23) or directly from (3.21) one can work out the formal series to any desired order. One finds the structure

$$m_p(t) = \sum_{n \geq p/2 \bmod 2} c_{p,n} (-4\alpha)^{\frac{n-p/2}{2}} \frac{(2t)^n}{n!}, \quad c_{p,n} \in \mathbb{N}. \quad (3.25)$$

Inserted into (3.23) this gives the recursion relations

$$\begin{aligned} c_{p,p/2} &= \frac{p(p-1)}{2} c_{p-2,p/2-1}, \\ c_{p,n} &= -(p-3)c_{p-4,n} + \frac{2}{p-2} c_{p-2,n+1}, \quad n > p/2. \end{aligned} \quad (3.26)$$

In particular $c_{p,p/2} = 2^{-p/2}p!$, so that $m_p(t) = p! t^{p/2}/(p/2)! + O(t^{p/2+1})$. The coefficients relevant for the second and fourth moment are related by $c_{4,n} = c_{2,n+1}$, and the first few are listed in Table 3, later on the first nonzero 500 are used.

$c_{2,1}$	1
$c_{2,3} = c_{4,2}$	6
$c_{2,5} = c_{4,4}$	216
$c_{2,7} = c_{4,6}$	22896
$c_{2,9} = c_{4,8}$	5360256
$c_{2,11} = c_{4,10}$	2346299136

Table 3: Coefficients in the asymptotic series (3.25) for $p = 2, 4$.

According to the Lemma the series (3.25) should be asymptotic to the exact result for $t \rightarrow 0$. By direct summation of partial sums up to $O(t^N)$ one finds that the result is N independent for small $0 \leq t < t_*(p)$, where $t_*(p) \approx 0.13$ for $p = 2, 4$. For small t the series thus defines a function whose values can be compared with the numerical simulations. One finds an excellent agreement

$$m_p(t)|_{\text{series}} = m_p(t)|_{\text{simulation}}, \quad 0 \leq t < t_*(p), \quad p = 2, 4. \quad (3.27)$$

This corroborates the Lemma and more specifically determines an interval $0 \leq t < t_*(p)$ in which the series (3.25) provides a valid description of the exact functions $m_p(t)$ in (3.22). Assuming that $\inf_p t_*(p) > 0$, this also leads to a strategy to prove the Parisi-Klauder conjecture for short times based on the series (3.25).

Next consider the growth rate of the coefficients. With the parameterization

$$\log \frac{c_{p,n}}{n!} = \alpha_p(k + 1/2) \ln k - \beta_p k, \quad n = p/2 - 2 + 2k, \quad k \in \mathbb{N}, \quad (3.28)$$

one finds α_p, β_p close to unity for $p = 2, 4$ based on the $c_{p,n}$ with $p/2 \leq n \leq 1000$. A proof for all n can be based on the recursion relations (3.26). Since $\ln n! \sim (n + 1/2) \ln n - n$ this in combination with the alternating signs indicates that Borel resummation techniques are applicable. In choosing the parameterization (3.28) we attributed powerlike terms to a redefinition of t via $t \mapsto (4\sqrt{\alpha}t)^2$.

With this understanding the conventional Borel sum rather than the Borel-Leroy generalization is applicable. We therefore define

$$\begin{aligned} b_p(s) &:= \sum_{n \geq p/2 \bmod 2} c_{p,n} (-4\alpha)^{\frac{n-p/2}{2}} \frac{(2s)^n}{n!^2}, \\ M_p(t) &:= t^{-1} \int_0^\infty ds e^{-s/t} b_p(s), \end{aligned} \quad (3.29)$$

initially for real $\alpha > 0$. One then finds that the partial sums $0 \leq n \leq 1000$ produce truncation independent results for the Borel sums b_2, b_4 in the interval $0 \leq s \leq 11$, as shown in the inserts of Figures 2,3. The restricted integration produces a well-defined Borel transform in the interval $0 \leq t \leq 2$, shown in Figures 2,3 (solid lines). The Borel transforms extend the low t regime of the original functions in (3.27). and quickly approach constant values at around $t = 1$. Importantly they also agree with the averages defined by the complex propagation kernel:

$$M_p(t) = \langle x^p \rangle_{\rho_t} \quad \text{for all } t \geq 0. \quad (3.30)$$

For $\alpha > 0$ the equality follows from the uniqueness of the Borel transform subject to (3.28). The extension to complex couplings can be done via scaling relations. Generally (3.29) implies

$$M_p(t) = \alpha^{-p/4} M_p|_{\alpha=1}(\alpha^{1/2}t), \quad (3.31)$$

provided $M_p(t)$ is analytic in $0 \leq \arg t < \pi/4$. For the Borel sum analyticity in $0 \leq \arg s < \pi/4$ is manifest and allows one to rewrite (3.31) in a way such that only the Borel sum for real arguments enters

$$M_p(t) = e^{-i\frac{\theta}{4}(1+\frac{p}{2})} t^{-1} \int_0^\infty ds \exp\left(-\frac{s}{te^{i\theta/4}}\right) b_p(s), \quad (3.32)$$

where $b_p(s)$ is the Borel sum for $\alpha > 0$ and $M_p(t)$ represents the moments for $e^{i\theta/2}\alpha$. On the other hand (3.31) matches precisely the scaling relation (2.34) derived in Section 2 from the properties of the complex propagation kernel. This shows (3.30) for all $0 \leq \theta < \pi$.

In contrast to the setting in Section 2 the Borel transform (3.32) allows for a direct computational implementation. The relation (3.32) therefore provides an *alternative* to the complex Langevin method in the case at hand. It directly gives the *time dependent* moments for (2.1) which we will compare in Section 4 with the results obtained from the complex Langevin simulations.

A compelling numerical demonstration of (3.30) for $\alpha > 0$ is obtained by comparing the results of a 1d Langevin simulation for $\langle x^p \rangle_{\rho_t}$, $\alpha > 0$ fixed, with the corresponding Borel transforms. The results are shown for $p = 2, 4$ in Figures 2,3. One also sees that the asymptotic values are approached quickly $\lim_{t \rightarrow \infty} \langle x^2 \rangle_{\rho_t} = \langle x^2 \rangle_{e^{-s}} = 0.47798$ and $\lim_{t \rightarrow \infty} \langle x^4 \rangle_{\rho_t} = \langle x^4 \rangle_{e^{-s}} = 0.5$, respectively.

Finally, the result (3.30) is also consistent with the recursion relation (3.23) remaining valid for all t with the $m_p(t) = \langle x^p \rangle_{\rho_t}$ interpreted as the Borel transform $M_p(t)$. Stationary distributions have $\partial_t^k m_2 = 0$, $k \geq 1$, and (3.23) simplifies to $4\alpha m_p(\infty) = (p-3)m_{p-4}(\infty)$,

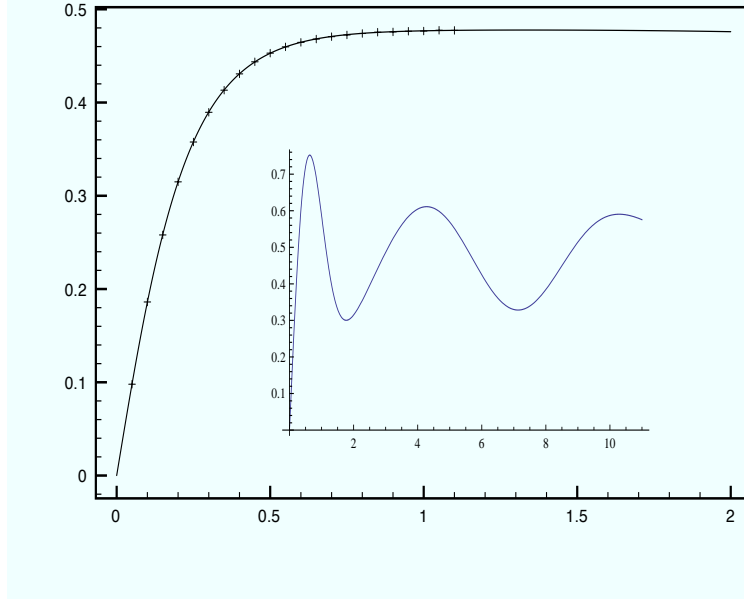


Figure 2: Results from 1d Langevin simulation at $\theta = 0, \lambda = 1$ for $m_2(t)$ versus the Borel transform $M_2(t)$. The insert shows the Borel sum $b_2(s)$.

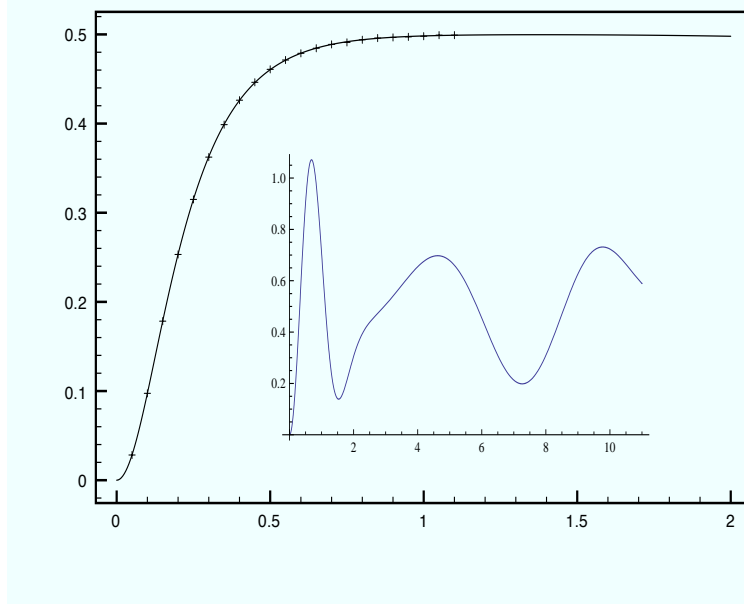


Figure 3: Results from the 1d Langevin simulation at $\theta = 0, \lambda = 1$ for $m_4(t)$ versus the Borel transform $M_4(t)$. The insert shows the Borel sum $b_4(s)$.

1

which is readily solved:

$$\begin{aligned}
 m_p(\infty) &= \frac{\Gamma(\frac{p+1}{4})}{\Gamma(\frac{1}{4})} \alpha^{-p/4}, & p &= 0 \pmod{4}, \\
 m_p(\infty) &= \frac{\Gamma(\frac{p+1}{4})}{\Gamma(\frac{3}{4})} m_2(\infty) \alpha^{(2-p)/4}, & p &= 2 \pmod{4}.
 \end{aligned} \tag{3.33}$$

This holds irrespective of the validity of the conjecture (3.1) or its $t \rightarrow \infty$ limiting form (3.6). Assuming that the $\langle \mathcal{O} \rangle_{\rho_t}$ evolution equation in (3.17) remains valid for all t and p the asymptotic values must obey (3.33). The limit $m_2(\infty)$ remains undetermined in agreement with the structure in (3.24). A specific choice for $m_2(\infty)$ will render (3.33) compatible with the asymptotic form of (1.8): the limit $\lim_{t \rightarrow \infty} \langle x^p \rangle_{\rho_t} = \langle x^p \rangle_{e-s}$ is of course trivially evaluated and gives

$$\langle x^p \rangle_{e-s} = \frac{\Gamma\left(\frac{p+1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \alpha^{-p/4}, \quad p = 0 \pmod{2}, \quad \text{Re} \alpha > 0. \quad (3.34)$$

Numerically, $\langle x^2 \rangle_{e-s} = 0.47798 \lambda^{-1/4}$, $\langle x^4 \rangle_{e-s} = 0.5 \lambda^{-1/4}$, $\langle x^6 \rangle_{e-s} = 0.7169 \lambda^{-3/4}$. One sees that (3.34) matches (3.33) iff $m_2(\infty) = \alpha^{-1/4} \Gamma(3/4)/\Gamma(1/4)$. Implicitly therefore, the Borel resummation described before fixes the parameter $m_2(\infty)$ undetermined by (3.33) to precisely this value.

In the $m_p(t) = \langle z^p \rangle_{R_t}$ interpretation of the moments the flow equation for $\langle \mathcal{O} \rangle_{R_t}$ in (3.17) might likewise be valid for all t but with a $m_2(\infty)$ value different from the one above. If so, at least the $m_p(\infty)$, $p = 0 \pmod{4}$, and the ratios $m_p(\infty)/m_2(\infty)$, $p = 2 \pmod{4}$, must come out as in (3.33). In particular $m_4(\infty) = 0.3536 \lambda^{-1/2} e^{-i\theta/2}$.

4. Complex Langevin Simulations: Method and Results

The complex Langevin process underlying (1.6) is realized as a Wiener evolution in the real two dimensional space corresponding to the real and imaginary parts of the integration variable. Specifically, for a single integration variable with complexification $z = x + iy$, one generates an ensemble of points in the (x, y) plane by integrating the stochastic pair of equations

$$\begin{aligned} \frac{dx}{dt} &= -\text{Re}\left(\frac{\partial S(x + iy)}{\partial x}\right) + b_R(t), \\ \frac{dy}{dt} &= -\text{Im}\left(\frac{\partial S(x + iy)}{\partial x}\right) + b_I(t), \end{aligned} \quad (4.1)$$

where b_R, b_I are independent Wiener (“white noise”) processes, with

$$\begin{aligned} \overline{b_R(t_1) b_R(t_2)} &= 2A_R \delta(t_1 - t_2), \\ \overline{b_I(t_1) b_I(t_2)} &= 2A_I \delta(t_1 - t_2), \end{aligned} \quad (4.2)$$

and $A_R = A_I + 1$ (complex fluctuation-dissipation relation). In practice, we realize this process by discretizing the time:

$$\begin{aligned} x(t + \Delta t) &= x(t) - \operatorname{Re}\left(\frac{\partial S(x + iy)}{\partial x}\right)\Delta t + \Delta b_R, \\ y(t + \Delta t) &= y(t) - \operatorname{Im}\left(\frac{\partial S(x + iy)}{\partial x}\right)\Delta t + \Delta b_I, \end{aligned} \quad (4.3)$$

where Δb_R (resp. Δb_I) are Gaussian distributed randoms of variance $2A_R\Delta t$ (resp. $2A_I\Delta t$). Two important aspects of our numerical implementation are (see also [13]):

1. We use continuously (i.e. normally) distributed random variables, rather than discrete random step variables, to avoid discretization artifacts in the generated ensemble. Thus, our ensemble values (starting at the origin $x = y = 0$, say) are not “quantized” but fill out a continuous region in the x - y configuration space.
2. More importantly, we have found that the Brownian process occasionally wanders into regions where the “force” functions $F_x = \operatorname{Re}(\frac{\partial S(x+iy)}{\partial x})$, $F_y = \operatorname{Im}(\frac{\partial S(x+iy)}{\partial x})$ become quite large. To avoid losing accuracy in the discretized realization (4.3), we therefore readjust the time step Δt at every update to ensure that the variations $\Delta x, \Delta y$ remain small. In practice, this is done by choosing

$$\Delta t = \frac{\delta}{1 + |F_x| + |F_y|} \quad (4.4)$$

with the nominal time step δ chosen at some suitably small value (typically, 10^{-5}). If one uses a fixed time step, we have found that it is quite common to obtain results which appear to converge, but are simply incorrect as a consequence of rare excursions which distort the result due to a loss of accuracy in the discretization of the Wiener process.

We have carried out an exhaustive numerical investigation of the Langevin time dependence for both real and complex polynomial actions, specifically for the quadratic (“free”) $S(z) = \omega e^{i\theta/2} z^2$, $\omega > 0, 0 \leq \theta < \pi$, detailed in Appendix A, and the quartic (“interacting”) $S(z) = \frac{1}{2}\sqrt{\lambda} e^{i\theta/2} z^4$, $\lambda > 0, 0 \leq \theta < \pi$, case. The Langevin time dependence of the second and fourth moments $\mathcal{O}(z) = z^2, z^4$ was studied by generating a large number (typically 1-4 million) of independent trajectories in the x - y plane (starting at the origin), using the adaptive discretized algorithm described above, and terminating each trajectory when the desired Langevin time was reached. This allowed us to study both the short time and long time asymptotics of the complex Langevin process.

For definiteness we present the results for the second moments. The behavior of the fourth moments is qualitatively similar but the breakdown times are often signalled by a sudden increase in fluctuations size. Figure 4 shows a comparison of the results from the Langevin

simulation of the second moments with those from the Borel transform (3.32) for several values of A_I and fixed $\theta = \pi/2$. The statistical errors for the Langevin results are smaller than the size of the symbols. The putative equilibrium values for large t coincide with those obtained by Aarts [14].

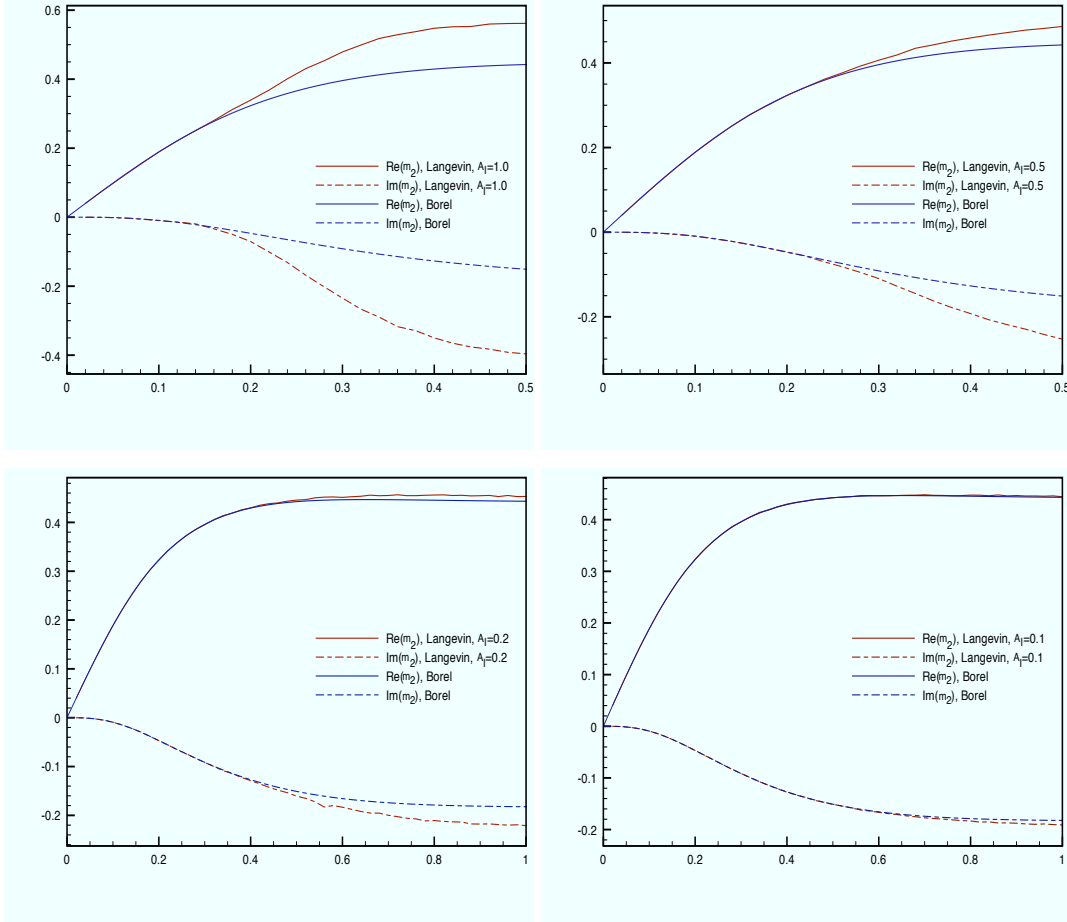


Figure 4: Second moments from Langevin simulations (red) for various values of A_I and $\theta = \pi/2$ compared to the Borel transform (blue).

One sees that in each case there is good agreement for $0 \leq t \leq t_c(A_I)$ but disagreement for $t > t_c(A_I)$. For the time being we define the ‘breakdown time’ $t_c(A_I)$ informally as the smallest time at which both results deviate significantly with respect to the (statistical and systematic) errors. This rests on the ‘experimental’ fact that the transition is relatively sharp; a theoretical understanding of this phenomenon is currently lacking. Assuming there is a sharp transition $t_c(A_I)$ visibly increases with decreasing A_I . For the values considered one has $t_c \approx 0.16, 0.22, 0.41, 0.67$ for $A_I = 1, 0.5, 0.2, 0.1$, respectively. Tentatively this suggests a powerlike scaling

$$t_c(A_I) \sim (A_I + \alpha)^{-\gamma}, \quad \alpha \approx 0, \quad \gamma \approx 0.6, \quad (4.5)$$

though other functional forms are compatible with the data as well. The precise form of the A_I dependence can of course not be pinned down numerically and neither can be the limit $\lim_{A_I \rightarrow 0} t_c(A_I)$.

However, the breakdown of the agreement for $A_I > 0$ shows that at best the refined version (1.13) of the conjecture can hold true. A similar conclusion was reached along different lines in [12]. In much of the early literature $A_I = 0$ was used by default. On the other hand the heuristic derivations of (1.8) do not discriminate between real and complex noise, and neither does the framework of [9, 10].

Although the Langevin results for $A_I > 0$ differ for $t > t_c(A_I)$ from the ‘desired’ results, they seem likewise to converge to finite asymptotic values $m_p(\infty)|_{\text{Langevin}}$. In line with the general setting described in Section 3.1 this should equal the average as computed with a positive equilibrium distribution φ_0 in the kernel of \mathbb{P} , see (3.15). In the next section we compute φ_0 directly from the defining relation $\mathbb{P}\varphi_0 = 0$ and verify that $\lim_{t \rightarrow \infty} \langle z^p \rangle_{R_t} = \langle z^p \rangle_{\varphi_0}$ indeed holds for the second moments considered.

5. Spectrum and ground state of \mathbb{P}

As stressed before the non-naive action of the semi-group generated by the real Fokker-Planck operator \mathbb{P} on holomorphic functions is the likely culprit for the failure of the method. Of course one has very little analytic control over the semigroup in question, so one might doubt that \mathbb{P} generates a well-defined semigroup at all. In the following we present numerical results indicating that at least several necessary conditions for this to be the case are satisfied. Along the way we also obtain approximations to the ground state wave function of \mathbb{P} which allows us to independently compute the relevant moments.

We continue to treat the anharmonic case as a paradigm, $S(z) = \frac{1}{2}\sqrt{\lambda}e^{i\theta/2}z^4$. The real Fokker-Planck operator \mathbb{P} in this case reads

$$\begin{aligned}\mathbb{P} &= A_R \partial_x^2 + A_I \partial_y^2 - F_x \partial_x - F_y \partial_y - \partial_x F_x - \partial_y F_y, \\ F_x &= -2\sqrt{\lambda} \left[(x^3 - 3xy^2) \cos \frac{\theta}{2} + (-3x^2y + y^3) \sin \frac{\theta}{2} \right], \\ F_y &= -2\sqrt{\lambda} \left[(3x^2y - y^3) \cos \frac{\theta}{2} + (x^3 - 3xy^2) \sin \frac{\theta}{2} \right],\end{aligned}\tag{5.1}$$

with $\partial_x F_y + \partial_y F_x = 0$. The operator has two manifest symmetries

$$\mathbb{P}|_{x \rightarrow \lambda^{-1/8}x, y \rightarrow \lambda^{-1/8}y} = \lambda^{1/4} \mathbb{P}|_{\lambda=1}, \quad \mathbb{P}|_{x \rightarrow -x, y \rightarrow -y} = \mathbb{P},\tag{5.2}$$

which constrain the structure of the spectrum and the eigenfunctions. The first one implies that the exact eigenvalues of \mathbb{P} scale like $\lambda^{1/4}$, just as they do for the complex Fokker-Planck

operator \mathbf{P} , which in turn follows from (2.17) and the fact that \mathbf{H} and \mathbf{P} are similar. Since \mathbb{P} is real for all parameter values the complex conjugate of an eigenfunction will again be an eigenfunction with the complex conjugate spectral value. This means the spectrum of \mathbb{P} lies symmetric to the real axis. It is plausibly purely discrete and the numerical results presented below are compatible with the structure (3.11), where $\lambda^{-1/4}E_n$ is λ -independent and E_0 is non-degenerate. Concerning the eigenfunctions $\mathbb{P}\varphi_n = E_n\varphi_n$, the scaling relation entails that they can be written as a λ -independent function evaluated at arguments $\lambda^{1/8}x, \lambda^{1/8}y$,

$$\varphi_n(x, y) = \varphi_n|_{\lambda=1}(\lambda^{1/8}x, \lambda^{1/8}y). \quad (5.3)$$

The reflection symmetry $\mathbb{P}|_{x \rightarrow -x, y \rightarrow -y} = \mathbb{P}$ entails that the eigenfunctions can be chosen to have definite parity, $\varphi_n(-x, -y) = \delta_n \varphi_n(x, y)$, $\delta_n \in \{\pm 1\}$. Since we expect the ground state to be positive it must be invariant

$$\varphi_0(x, y) = \varphi_0(-x, -y), \quad (5.4)$$

which also ensures that odd observables $\mathcal{O}(x + iy)$ have vanishing expectation value (3.7).

5.1 Spectrum

In a first step we now investigate the spectrum of \mathbb{P} using a variant of the matrix truncation technique from Section 2. Anticipating that zero lies in the spectrum the associated eigenvector can be used to approximate φ_0 and to compute the required averages. To this end it is useful to choose real basis functions like the products $H_k(x)H_l(y)$ of Hermite functions in x and y . Reality of the approximate φ_0 can then be seen directly and only the test for positivity requires further analysis. The price to pay is that the matrix elements in a real basis are more complicated than in a complex basis adapted to (3.2). We introduce a pair of oscillators $[a, a^*] = 1 = [b, b^*]$ by

$$\begin{aligned} x &= \frac{1}{\sqrt{2\omega}}(a^* + a), & p_x &= i\sqrt{\frac{\omega}{2}}(a^* - a) = -i\partial_x, \\ y &= \frac{1}{\sqrt{2\omega}}(b^* + b), & p_y &= i\sqrt{\frac{\omega}{2}}(b^* - b) = -i\partial_y, \end{aligned} \quad (5.5)$$

with $\omega := \sqrt{6}\lambda^{1/4}$. Further, we decompose \mathbb{P} as follows

$$\begin{aligned} -\mathbb{P} &= A_R p_x^2 + A_I p_y^2 + 2\omega^2[(y^2 - x^2) \cos \frac{\theta}{2} + 2xy \sin \frac{\theta}{2}] \\ &+ \frac{\omega}{12} \cos \frac{\theta}{2} [X(x, y) - X(y, x)] + \frac{\omega}{12} \sin \frac{\theta}{2} [Y(x, y) + Y(y, x)], \end{aligned} \quad (5.6)$$

where

$$X(x, y) = -i4\omega(x^3 - 3xy^2)p_x, \quad Y(x, y) = -i4\omega(x^3 - 3xy^2)p_y. \quad (5.7)$$

The symmetry (5.2) now amounts to $\mathbb{P}|_{x \rightarrow x/\sqrt{\omega}, y \rightarrow y/\sqrt{\omega}} = \omega \mathbb{P}|_{\omega=1}$, so that ω drops out upon insertion of (5.5). One finds in terms of the oscillators

$$\begin{aligned} -\frac{2}{\omega} \mathbb{P} &= (A_R - 2 \cos \frac{\theta}{2})(2a^*a + 1) + (A_I + 2 \cos \frac{\theta}{2})(2b^*b + 1) \\ &- (A_R + 2 \cos \frac{\theta}{2})(a^{*2} + a^2) + (-A_I + 2 \cos \frac{\theta}{2})(b^{*2} + b^2) + 4 \sin \frac{\theta}{2}(a^* + a)(b^* + b) \\ &+ \frac{1}{6} \cos \frac{\theta}{2} [X(a, b) - X(b, a)] + \frac{1}{6} \sin \frac{\theta}{2} [Y(a, b) + Y(b, a)], \end{aligned}$$

where

$$\begin{aligned} X(a, b) &= a^{*4} - a^4 + 2a^{*3}a - 2a^*a^3 + 6a^{*2} + 6a^*a + 3 - 3(b^{*2} + 2b^*b + b^2 + 1)(a^{*2} - a^2 + 1), \\ Y(a, b) &= (a^{*3} + 3a^{*2}a + 3a^*a^2 + a^3 + 3a^* + 3a)(b^* - b) \\ &- 3(a^* + a)(b^{*3} + b^{*2}b - b^*b^2 - b^3 + 3b^* + b). \end{aligned} \quad (5.8)$$

The basis of Hermite functions in $\sqrt{\omega}x$, $\sqrt{\omega}y$ corresponds to

$$|m, n\rangle := \frac{1}{\sqrt{m!n!}} a^{*m} b^{*n} |0\rangle, \quad a|0\rangle = b|0\rangle = 0. \quad (5.9)$$

Finally the matrix elements of (5.8) come out as

$$\begin{aligned} -\frac{2}{\omega} \langle kl | \mathbb{P} | mn \rangle &= \left[(A_R - 2 \cos \frac{\theta}{2})(2k + 1) + (A_I + 2 \cos \frac{\theta}{2})(2l + 1) \right] \delta_{km} \delta_{ln} \\ &- (A_R + 2 \cos \frac{\theta}{2}) [\sqrt{k(k-1)} \delta_{k-2,m} + \sqrt{m(m-1)} \delta_{m-2,k}] \delta_{l,n} \\ &+ (-A_I + 2 \cos \frac{\theta}{2}) [\sqrt{l(l-1)} \delta_{l-2,n} + \sqrt{n(n-1)} \delta_{n-2,l}] \delta_{k,m} \\ &+ 4 \sin \frac{\theta}{2} (\sqrt{k} \delta_{k-1,m} + \sqrt{m} \delta_{m-1,k}) (\sqrt{l} \delta_{l-1,n} + \sqrt{n} \delta_{n-1,l}) \\ &+ \frac{1}{6} \cos \frac{\theta}{2} [X_{kl,mn} - X_{lk,nm}] + \frac{1}{6} \sin \frac{\theta}{2} [Y_{kl,mn} + Y_{lk,nm}], \end{aligned} \quad (5.10)$$

where

$$\begin{aligned}
X_{kl,mn} = & \delta_{l,n} \sqrt{\frac{k!}{m!}} [\delta_{k,m+4} + 2(k-2)\delta_{k,m+2}] - \delta_{l,n} \sqrt{\frac{m!}{k!}} [\delta_{m,k+4} + 2(m-2)\delta_{m,k+2}] \\
& + \delta_{ln} [6\sqrt{k(k-1)}\delta_{k,m+2} + (6k+3)\delta_{k,m}] \\
& - 3[\sqrt{(l+2)(l+1)}\delta_{l+2,n} + \sqrt{l(l-1)}\delta_{l-2,n} + (2l+1)\delta_{l,n}] \\
& \times [\sqrt{(m+2)(m+1)}\delta_{k,m+2} - \sqrt{m(m-1)}\delta_{k,m-2} + \delta_{k,m}] \quad (5.11a)
\end{aligned}$$

$$\begin{aligned}
Y_{kl,mn} = & [\sqrt{l}\delta_{l-1,n} - \sqrt{l+1}\delta_{l+1,n}] [\sqrt{(m+3)(m+2)(m+1)}\delta_{m+3,k} \\
& + 3\sqrt{m+1}(m+1)\delta_{m+1,k} + 3\sqrt{m}m\delta_{m-1,k} + \sqrt{m(m-1)(m-2)}\delta_{m-3,k}] \\
& - 3[\sqrt{k}\delta_{k-1,m} + \sqrt{k+1}\delta_{k+1,m}] [\sqrt{(n+3)(n+2)(n+1)}\delta_{l,n+3} \\
& + \sqrt{n+1}(n+3)\delta_{l,n+1} - \sqrt{n}(n-2)\delta_{l,n-1} - \sqrt{n(n-1)(n-2)}\delta_{l,n-3}]. \quad (5.11b)
\end{aligned}$$

The matrix elements of the transpose operator $\mathbb{P}^T = A_R\partial_x^2 + A_I\partial_y^2 + F_x\partial_x + F_y\partial_y$, are obtained similarly and read

$$\begin{aligned}
-\frac{2}{\omega} \langle kl | \mathbb{P}^T | mn \rangle = & [A_R(2k+1) + A_I(2l+1)] \delta_{km} \delta_{ln} \\
& - A_R [\sqrt{k(k-1)}\delta_{k-2,m} + \sqrt{m(m-1)}\delta_{m-2,k}] \delta_{l,n} \\
& - A_I [\sqrt{l(l-1)}\delta_{l-2,n} + \sqrt{n(n-1)}\delta_{n-2,l}] \delta_{k,m} \\
& - \frac{1}{6} \cos \frac{\theta}{2} [X_{kl,mn} - X_{lk,nm}] - \frac{1}{6} \sin \frac{\theta}{2} [Y_{kl,mn} + Y_{lk,nm}]. \quad (5.12)
\end{aligned}$$

One can check

$$\langle kl | \mathbb{P}^T | mn \rangle = \langle mn | \mathbb{P} | kl \rangle, \quad (5.13)$$

so for the matrix truncations the last spectral equality in (3.13) is manifestly satisfied.

The numerical computation of the spectrum is now straightforward. After conversion of the double index $k, l = 0, \dots, N-1$ into a single via $i = kN + l + 1$, the resulting matrix $P_{ij} := -(2/\omega) \langle kl | \mathbb{P} | mn \rangle$, $i, j = 1, \dots, N^2$, can be diagonalized numerically. The reliability and accuracy of the approximate eigenvalues can be assessed by increasing N . Since some of the eigenvalues are complex the ordering is done by modulus, i.e. $n > m$ if $|E_n| > |E_m|$. With a C code implementing the Hessenberg transformation running times are 1–6 hours for $N = 80-100$ on a conventional laptop with effectively 2G RAM usage. In order to reach $N = 160$ without cumbersome recoding we used a 16G RAM workstation with running

times 8 – 140 hours. The approximate spectra stabilize convincingly at the $O(10^{-3})$ level, from the trend upon varying N between 100 – 160 we guesstimated a systematic error. For illustration we report the results for a specific case: $A_I = 1, \theta = 0, \pi/2$ in Table 4. By the argument after Eq. (3.12) the lowest eigenvalue must be zero.

n	$A_R = 2, A_I = 1, \theta = 0$	$A_R = 2, A_I = 1, \theta = \pi/2$
0	0.002(2)	0.006(6)
1	2.0725(2)	1.921(2)
2	5.4920(2)	5.833(2)
3	8.5377(2)	7.885(2)
4	$8.314(3) \pm i 5.798(3)$	$8.590(3) \pm i 6.092(3)$
5	$13.962(3) \pm i 5.998(3)$	$13.125(3) \pm i 5.254(3)$
6	$14.310(3) \pm i 8.346(3)$	$15.074(3) \pm i 9.160(3)$

Table 4: Low lying eigenvalues for \mathbb{P} with $\lambda = 1$ based on $N = 100 - 160$ truncations.

One sees that the first few eigenvalues are real while higher excited states typically come in complex conjugate pairs. Importantly the real parts are nonnegative so that the spectrum is compatible with \mathbb{P} being the generator of a semigroup with real kernel. The spectrum for other parameter values was found to be qualitatively similar and is compatible with (3.11). Positivity of the kernel is essential for the interpretation of R_t as a probability measure but is not manifest from the spectrum.

5.2 Ground state

In contrast to the quadratic case the determination of the ground state wave function is nontrivial. First note that now $\exp\{-S(x + iy)\}$ is not even an eigenfunction of \mathbb{P} . This is because although ∂_y acts like $i\partial_x$ on holomorphic functions, the replacement is illegitimate in the $\partial_y F_y$ term. This results in $e^{+S}\mathbb{P}e^{-S} = -i\text{Im}\partial_x^2 S - \partial_y F_y = 6\sqrt{\lambda}e^{-i\theta/2}(x - iy)^2$. The special real solution (A.17) generalizes to the interacting case but is again not integrable. For generic A_R it is tempting to search for elements in the kernel of the form

$$\varphi_0(x, y) = \exp\{-A_0 x^4 - 3B_0 x^3 y - 6C_0 x^2 y^2 - 3D_0 x y^3 - E_0 y^4\}. \quad (5.14)$$

By direct computation one sees that no such solution exists for $A_R \neq 1/2$. Since $\mathbb{P}\varphi_0 = E_0\varphi_0$ is an elliptic second order differential equation one expects the existence of a solution to be determined by general principles. The usual existence theorems however refer to bounded domains with Dirichlet boundary conditions. In the situation at hand compactification of \mathbb{R}^2 to the unit square, say, introduces singularities in the coefficient functions of \mathbb{P} towards the boundary where the solution is supposed to vanish. In the interior the desired solution must be non-negative. Establishing (non-)existence of a solution along these lines therefore may be non-trivial.

Using the matrix truncation technique the numerical determination of the ground state is however straightforward. Once the ground state energy E_0 is reliably identified at the chosen truncation level N one can compute the associated eigenvector of P_{ij} to high accuracy by iterated action of the resolvent. The resulting vector v_i , $i = 1, \dots, N^2$ is converted into the approximate ground state wave function by the transformation

$$\varphi_0(x, y) = \sum_{i=1}^{N^2} v_i H_{k(i)}(\sqrt{\omega}x) H_{l(i)}(\sqrt{\omega}y), \quad (5.15)$$

where the ω -dependence is restored and $k(i) = (i - 1 - \text{mod}(i - 1, N))/N$, $l(i) = \text{mod}(i - 1, N)$ converts the indices, with $\text{mod}(m, n)$ defined as m modulo n . The required basis of Hermite functions is conveniently programmed using the recursion relation. As a test on the accuracy of the conversion one can check $\int dx dy \varphi_0(x, y)^2 = \omega^{-1} \sum_i v_i^2$.

We computed the eigenvector v_i , $i = 1, \dots, N^2$, for the same parameters sets as in Table 4 and converted it into the corresponding ground state wave function φ_0 using (5.15). The results are shown in Figure 6. It is numerically nontrivial that they come out strictly positive and therefore can serve as a probability measure.

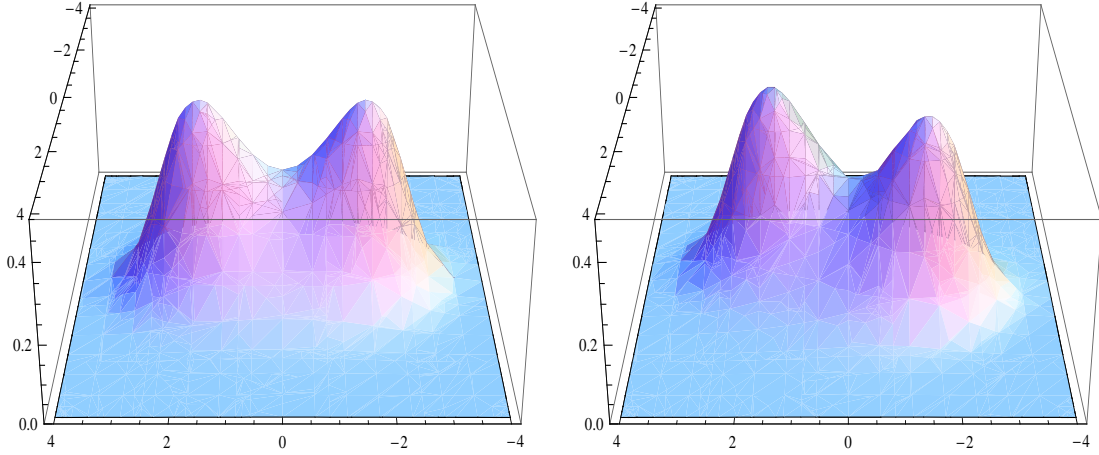


Figure 5: Ground state wave function for \mathbb{P} for truncation $N = 140$ and parameters $A_R = 2, A_I = 1$. Left $\theta = 0$, right $\theta = \pi/2$.

For nonzero θ one sees that the principal axes in Figure 5 is rotated by an angle $\theta/4$. By analogy with the non-interacting case described in Appendix B one should not expect, however, that φ_0 differs from $\varphi_0|_{\theta=0}$ only by a rigid rotation. For other values of A_I the results for the ground state wave functions are qualitatively similar: a double peak is visible whose principal axes is rotated by an angle $\theta/4$. The main effect of decreasing A_I is to shrink the extension of the level surfaces in the y -direction and to raise the summits. This is illustrated in Figure 6.

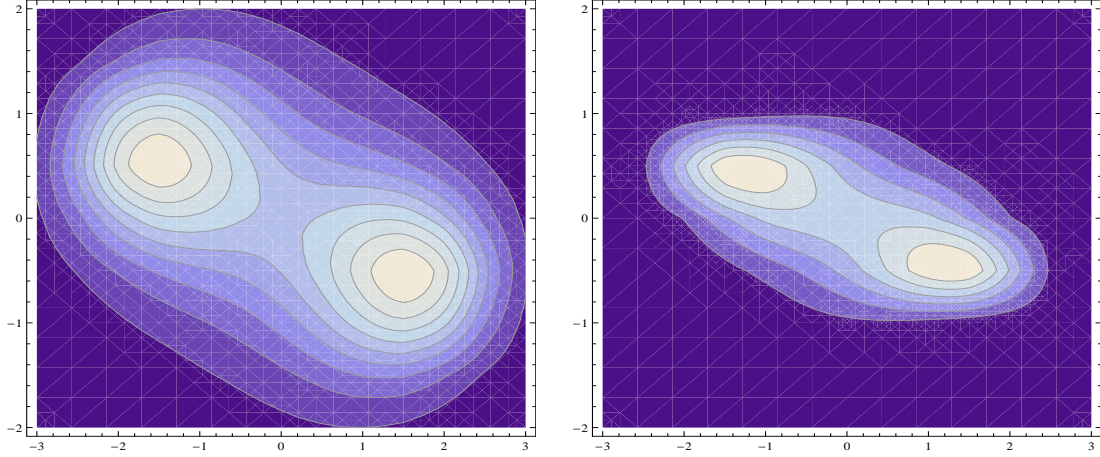


Figure 6: Level surfaces of $N = 140$ ground state wave functions for $\theta = \pi/2$. Left $A_I = 1$, right $A_I = 0.1$.

Finally we computed the second moments for the same parameter values as in the Langevin study. The Langevin moments for large times have previously been obtained by Aarts [14]; ours are in full agreement. Here we see that the latter also agree within the errors with those obtained from the directly computed putative equilibrium measure φ_0 .

A_I	1	0.5	0.2	0.1
$m_2(\varphi_0)$	$0.54 - i0.41$	$0.48 - i0.30$	$0.45 - i0.23$	$0.44 - i0.20$
$m_2(\infty)_L$	$0.56 - i0.41$	$0.49 - i0.29$	$0.45 - i0.23$	$0.44 - i0.20$

Table 5: Asymptotic second moments for $\theta = \pi/2$ as computed from the $N = 150$ ground state φ_0 and from the Langevin dynamics. Both agree within the errors but differ from the ‘desired’ one set by the complex measure: $m_2(\infty) = 0.441596 - i0.182915$.

Hence

$$\lim_{t \rightarrow \infty} \langle z^p \rangle_{R_t} = \langle z^p \rangle_{\varphi_0}, \quad \mathbb{P}\varphi_0 = 0, \quad \varphi_0 > 0, \quad (5.16)$$

holds for the second moments considered. This provides direct computational evidence that the interplay between the stochastic differential equations (4.1) and the semigroup generated by the real Fokker-Planck operator \mathbb{P} is indeed as expected. Further the $t \rightarrow \infty$ limit is compatible with the projection property onto the ground state φ_0 , as stipulated in (3.15). Nevertheless (3.6) fails for $A_I > 0$ which via (3.21) signals that property (iv) in Section 3.2 fails.

6. Conclusions

We critically reexamined the complex Langevin method in the paradigmatic case of a $e^{i\theta/2}\phi^4$ complex measure. In contrast to earlier studies we focussed on the temporal rather than the equilibrium aspects of the stochastic dynamics. Our main result is that the short time asymptotics encapsulates information also about the equilibrium aspects. Both sides of the conjectured identity (1.8) have identical asymptotic expansions around $t = 0$. The coefficients are such that the series is Borel summable and the Borel transform $M_p(t)$ of the moments' time series correctly captures the dynamics of the complex measure: $M_p(t) = \langle x^p \rangle_{\rho_t}$, for all $t \geq 0$. The resummation of the $t = 0$ asymptotic expansion therefore provides a practically usable alternative to the complex Langevin method, at least for low dimensional systems.

As to the validity of the conjecture itself, the results of sections 4 and 5 provide a counterexample for its validity for $A_I > 0$, in line with [14] and similar results in other systems [13]. At best the modified form (1.13) of the conjectured identity can therefore hold true. Even for this variant, however, counterexamples can be found. For example, for $S(x) = \frac{1}{2}\sqrt{\lambda}e^{i\theta/2}(x^2 \pm \omega^2)^2$ we find for $A_I = 0$ agreement of the putative equilibrium Langevin results with the analytical answer when the potential has a single well (plus sign) but not for a double well (minus sign) and $\theta \neq 0$. See [5] for related numerical results. In the $\omega^2 = 0$ case considered here the detailed analysis of the non-selfadjoint generators \mathbf{H} and \mathbb{P} (spectrum, spectral expansion, ground state) revealed no pathological features of the semigroups $e^{-t\mathbf{H}}$, $t > 0$, and $e^{t\mathbb{P}}$, $t > 0$. In particular both semigroups project for $t \rightarrow \infty$ correctly only the respective ground states, $e^{-S/2}$ and φ_0 . For $e^{-t\mathbf{H}}$, $t > 0$, the results of Section 2 come close to a mathematical proof thereof. The deviation from selfadjointness produces nontrivial spectral norms which impact the quasi-spectra and limit the extension to a holomorphic semigroup. For t real and large relevant in the present context, however, the $(e^{t\mathbb{P}})(x, x')$ convolutions converge to e^{-S} averages even for couplings where the potential is unbounded from below. For the $e^{t\mathbb{P}}$, $t > 0$, semigroup analytical control over the projection property is more difficult, but we regard the computational evidence in Section 5 as quite convincing.

Although both sides of (1.8) have well defined limits, for $A_I > 0$ the limits are different:

$$\langle x^P \rangle_{e^{-S}} = \lim_{t \rightarrow \infty} \langle x^p \rangle \neq \lim_{t \rightarrow \infty} \langle z^p \rangle_{Rt} = \langle z^p \rangle_{\varphi_0}. \quad (6.1)$$

We isolated as the culprit the limited validity of the identity $\int dx' dy' \exp(t\mathbb{P}^T)(x, 0; x', y') \mathcal{O}(x' + iy') = \int dx' \exp(t\mathbf{P}^T)(x, x') \mathcal{O}(x')$, for all $t > 0$ and all x . The identity holds by construction pointwise in x as an asymptotic series in t and both sides are termwise independent of A_I . For $t \rightarrow \infty$ the identity turns into (3.21) which fails for $A_I > 0$. It does so not because an integrations by part step fails but because the ground state φ_0 has the wrong structure. Since φ_0 is manifestly A_I dependent the A_I -independence of averages of holomorphic observables would be a nontrivial bonus property – which φ_0 simply fails to have.

Another way to look at the problem is in terms of the under-determination of the observable flow (3.17). The approach of [9, 10] aims at selecting the solution of the stationary equations corresponding to the complex Langevin process driven by \mathbb{P} via minimization of a suitable effective potential. However the one-parametric family of \mathbb{P} 's used will in general not produce results independent of the parameter. The parameter drops out in the short time asymptotics which provides one rationale for the temporal Borel resummation. When applicable, the Borel resummation augments the missing piece of information in a way compatible with certain analyticity properties in t and the coupling.

Compared to the field theoretical setting aimed at we ignored mass and kinetic terms. In line with earlier investigations [5, 8, 9, 10, 11] we regarded the analysis of a pure potential interaction as crucial. The inclusion of subleading mass and kinetic terms should not affect the qualitative aspects of the picture obtained. In particular it should be interesting to see whether the temporal Borel resummation technique extends to other systems.

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A: The Langevin method for quadratic complex actions

In the noninteracting case most of the constituents of the complex Langevin method can be computed explicitly, although previously only real noise seems to have been considered [7, 6]. Here we allow for $A_I > 0$ and show that the flow equations (3.17) uniquely determine the time dependent moments without the need of knowing the real and complex propagation kernels. The validity of the Parisi-Klauder conjecture for all $A_I \geq 0$ follows. The equilibrium distribution of the real measure can be obtained in closed form and the numerical evaluation of the equilibrium distribution via the complex Langevin simulation shows perfect agreement with the analytical result. For the complex propagation kernel we briefly review Davies' construction and present a generating function for the spectral norms.

We begin by describing the relation between the familiar Mehler kernel and the complex transfer operator with its nontrivial spectral norms. The hamiltonian is that of the complex harmonic oscillator,

$$\mathbf{H}_\theta = p^2 + \omega^2 e^{i\theta} q^2, \quad \omega > 0, \quad 0 \leq \theta < \pi, \quad (\text{A.1})$$

where the phase is again normalized so that it corresponds to the Wick rotation angle of the original problem. Note that for $\pi/2 < \theta < \pi$ the potential is unbounded from below. On account of a scaling argument (A.1) has spectrum $E_n = \omega e^{i\theta/2}(1 + 2n)$, $n \in \mathbb{N}_0$, with eigenfunctions $\Omega_n(q) \sim H_n(e^{i\theta/4}q)$, where the Hermite functions H_n are ortho-normalized with respect to the L^2 inner product. The complex eigenfunctions Ω_n also admit a Fock space description in terms of creation and annihilation operators

$$\begin{aligned} a_\theta &= e^{i\theta/4} \sqrt{\frac{\omega}{2}} q + i e^{-i\theta/4} \frac{1}{\sqrt{2\omega}} p = \cos \frac{\theta}{2} a + i \sin \frac{\theta}{2} a^*, \\ \bar{a}_\theta &= e^{i\theta/4} \sqrt{\frac{\omega}{2}} q - i e^{-i\theta/4} \frac{1}{\sqrt{2\omega}} p = \cos \frac{\theta}{2} a^* + i \sin \frac{\theta}{2} a, \end{aligned} \quad (\text{A.2})$$

where $[a_\theta, \bar{a}_\theta] = [a, a^*] = 1$. Since $a_\theta^* = \bar{a}_{-\theta}$ the L^2 adjoint does not preserve the commutation relations. Nevertheless $\mathbf{H}_\theta = \omega e^{i\theta}(2\bar{a}_\theta a_\theta + 1)$ holds and $|n\rangle := n!^{-1/2} \bar{a}_\theta^n |0\rangle$, $a_\theta |0\rangle = 0$, are the realization of the eigenfunctions Ω_n . Note that the (θ -dependent) $|0\rangle$ can be viewed as a coherent state over the $\theta = 0$ Fock vacuum $|0\rangle$. To define the relevant inner product we introduce a linear anti-involution ι on the algebra \mathcal{A} generated by the a_θ, \bar{a}_θ by

$$\iota(a_\theta) := \bar{a}_\theta, \quad \iota(zx) = z\iota(x), \quad \iota(xy) = \iota(y)\iota(x), \quad z \in \mathbb{C}, \quad x, y \in \mathcal{A}. \quad (\text{A.3})$$

Then $(x|0), y|0)_\theta := (|0), \iota(x)y|0)_\theta$ with $(|0), |0)_\theta = 1$ defines a \mathbb{R} -bilinear positive quadratic form over \mathcal{A} . In particular the Fock space realization of the Ω_n is an orthonormal basis with respect to $(\ , \)_\theta$. Returning to position space and the usual L^2 inner product $\langle \psi, \varphi \rangle = \int dq \psi(q)^* \varphi(q)$, this means the Ω_n^*, Ω_n form a bi-orthogonal basis in L^2 while the quantities $\langle \Omega_n, \Omega_n \rangle = N_n(\theta)$ are the spectral norms, see Eqs. (2.11) – (2.13). On account

of (A.2) they can in principle be computed algebraically. More conveniently a generating functional for the N_n can be obtained from the Mehler kernel. In the present conventions it reads

$$\begin{aligned} (e^{-\frac{t}{2}\mathbf{H}_0})(q, q') &= \frac{1}{\sqrt{2\pi \sinh t}} \exp \left\{ -\frac{q^2 + q'^2}{2 \tanh t} + \frac{qq'}{\sinh t} \right\} \\ &= \sum_{n \geq 0} e^{-t(1/2+n)} H_n(q) H_n(q'), \end{aligned} \quad (\text{A.4})$$

where we take $\omega = 1$ from now on. Pointwise the substitution $q \mapsto e^{i\theta/4}q, q' \mapsto e^{i\theta/4}q'$ is legitimate producing the transfer operator of the complex harmonic oscillator

$$(e^{-\frac{t}{2}\mathbf{H}_0})(e^{i\theta/4}q, e^{i\theta/4}q') = \sum_{n \geq 0} e^{-t(1/2+n)} P_n(q, q') = \exp \left\{ -\frac{t}{2} e^{-i\theta/2} \mathbf{H}_\theta \right\} (q, q'), \quad (\text{A.5})$$

with $P_n(q, q') = \Omega_n(q) \Omega_n(q')$. Assuming that for $q = q'$ integration and summation can be exchanged one obtains

$$\left(\frac{r}{(r + r^{-1}) \cos \frac{\theta}{2} - 2} \right)^{1/2} = \sum_{n \geq 0} r^{-n} N_n, \quad r = e^t. \quad (\text{A.6})$$

In particular

$$N_0 = \frac{1}{\cos^{1/2} \frac{\theta}{2}}, \quad N_1 = \frac{1}{\cos^{3/2} \frac{\theta}{2}}, \quad N_2 = \frac{5 - \cos \theta}{4 \cos^{5/2} \frac{\theta}{2}}, \quad \text{etc}, \quad (\text{A.7})$$

from which one also infers a powerlike divergence as $\theta \rightarrow \pi_-$. In addition one has the asymptotics

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln N_n = \gamma(\theta) < \infty, \quad (\text{A.8})$$

where an explicit expression for $\gamma(\theta)$ is known [20]. Norm convergence of the sum in (A.5) requires that $\frac{t}{2} \text{Re} E_n - \ln N_n$ is negative for large n , leading to the conclusion that

$$t > t_* = \frac{\gamma(\theta)}{\cos \frac{\theta}{2}}, \quad (\text{A.9})$$

suffices for norm convergence. In particular the $t \rightarrow \infty$ limit still projects onto the ground state, see (2.33). The semigroup $t \rightarrow e^{-t\mathbf{H}}$ in fact defines a bounded holomorphic semigroup for certain θ -dependent sectors of the complex t plane [21, 22]. In addition (A.3) affects resolvent estimates and the quasi-spectra [20, 22].

The hamiltonian (A.1) is up to an additive constant the Fokker-Planck hamiltonian for the complex action $S(q) = \omega e^{i\theta/2} q^2$. By construction it is up to a sign isospectral to the complex Langevin operator \mathbf{P} , viz

$$\begin{aligned}\mathbf{P} &= \partial_x^2 + 2\omega e^{i\theta/2} x \partial_x + 2\omega e^{i\theta/2}, \\ \mathbf{H} &= -e^{S(x)/2} \mathbf{P} e^{-S(x)/2} = -\partial_x^2 + \omega^2 e^{i\theta} x^2 - \omega,\end{aligned}\tag{A.10}$$

where we omit the subscript θ from now on. The real Langevin operator reads

$$\mathbb{P} = A_R \partial_x^2 + A_I \partial_y^2 + 2\omega \left(x \cos \frac{\theta}{2} - y \sin \frac{\theta}{2} \right) \partial_x + 2\omega \left(y \cos \frac{\theta}{2} + x \sin \frac{\theta}{2} \right) \partial_y + 4\omega \cos \frac{\theta}{2}. \tag{A.11}$$

The propagation kernel $t \mapsto e^{t\mathbf{P}}$ has been studied before in terms of \mathbf{H} . The propagation kernel $t \mapsto e^{t\mathbb{P}}$ is nontrivial even in the free case. Remarkably neither of these objects is needed to prove the Parisi-Klauder conjecture for the quadratic action.

The flow equations (3.17) for the powers (3.19) now assume the simple form $\partial_t m_p = p(p-1)m_{p-2} - 2\omega p m_p$, for $\theta = 0$, and the dependence on the phase can trivially be restored by rescaling $\omega \mapsto e^{i\theta/2}\omega$. These are decoupled differential equations with solution

$$m_p(t) = e^{-2p\omega t} m_p(0) + p(p-1) \int_0^t dt' m_{p-2}(t') e^{2\omega p(t'-t)}. \tag{A.12}$$

Since $m_0(t) = 1$, all moments are determined recursively up to their values at $t = 0$. Since by assumption the initial values in both interpretations (3.19) coincide, all moments coincide for all t , thereby verifying the Parisi-Klauder conjecture in this case. Alternatively, one can use the evolution $\partial_t \mathcal{O} = (\partial_z^2 - \partial_z \mathcal{O} \partial_z) \mathcal{O}$ to define ‘eigen-observables’ and reach the same conclusion. The eigenvalue equation translates into

$$(\partial_x^2 - 2\omega x \partial_x) \mathcal{O}_n = -2n\omega \mathcal{O}_n, \tag{A.13}$$

which is the defining relation for the n -th Hermite polynomial related by $\mathcal{O}_n = \Omega_n e^{-S/2}$ to the Hermite functions Ω_n . The flow equations (3.17) are trivially solved

$$\langle \mathcal{O}_n \rangle_{R_t} = e^{-2n\omega t} \langle \mathcal{O}_n \rangle_{R_0} = e^{-2n\omega t} \langle \mathcal{O}_n \rangle_{\rho_0} = \langle \mathcal{O}_n \rangle_{\rho_t}, \tag{A.14}$$

and are equivalent to (A.12).

Since (A.12) verifies the Parisi-Klauder conjecture one expects that its $t \rightarrow \infty$ limit verifies the asymptotic form of the conjecture. Indeed, for $t \rightarrow \infty$ the dependence on the initial values drops out in (A.12) and the limits are directly related by the recursion $2\omega m_p(\infty) = (p-1)m_{p-2}(\infty)$. Hence

$$m_p(\infty) = \frac{p!}{(p/2)!} \left(\frac{1}{4\omega} \right)^{p/2} = \langle x^p \rangle_{e^{-S}} = \langle z^p \rangle_{\varphi_0}, \tag{A.15}$$

where the last equality follows from the uniqueness of the solution (A.12), assuming only that φ_0 exists.

It is instructive to see that the equilibrium distribution φ_0 can in the harmonic case be found explicitly. We begin by searching for solutions of $\mathbb{P}\varphi_0 = E_0\varphi_0$ in the form

$$\varphi_0(x, y) = \exp\{-Ax^2 - 2Bxy - Cy^2\}. \quad (\text{A.16})$$

For generic $A_R = A_I + 1$ one finds three solutions. In addition to the expected

$$\varphi_0(x, y) = \exp -S(x + iy), \quad E_0 = 2\omega e^{-i\theta/2}, \quad (\text{A.17})$$

and its complex conjugate, there is a real solution with $E_0 = 0$:

$$\begin{aligned} A_0 &= \frac{2\omega \cos \frac{\theta}{2} (2 \cosh \frac{\alpha}{2} - 1 - \cos \theta)}{\cosh \alpha - \cos \theta} \\ B_0 &= \frac{\omega (\sin \frac{\theta}{2} + \sin \frac{3\theta}{2})}{\cosh \alpha - \cos \theta} \\ C_0 &= \frac{2\omega \cos \frac{\theta}{2} (2 \cosh \frac{\alpha}{2} + 1 + \cos \theta)}{\cosh \alpha - \cos \theta}, \end{aligned} \quad (\text{A.18})$$

where we set $A_R = \cosh^2 \alpha/4$, $A_I = \sinh^2 \alpha/4$. For $\alpha = 0$ this reduces to the result in [7, 6]. The solution becomes non-normalizable for $\theta \rightarrow \pi_-$ as $\varphi_0(x, y) \rightarrow 1$. After computing the averages, however, the limit $\theta \rightarrow \pi_-$ is well-defined, see (A.22) below. The coefficients in (A.18) are such that

$$\begin{pmatrix} A_0 & B_0 \\ B_0 & C_0 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{4} & \sin \frac{\theta}{4} \\ -\sin \frac{\theta}{4} & \cos \frac{\theta}{4} \end{pmatrix} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{4} & -\sin \frac{\theta}{4} \\ \sin \frac{\theta}{4} & \cos \frac{\theta}{4} \end{pmatrix}$$

$$\lambda_{\pm} = \frac{2\omega \cos \frac{\theta}{2}}{\cosh \frac{\alpha}{2} \pm \cos \frac{\theta}{2}}, \quad \frac{1}{\lambda_+} - \frac{1}{\lambda_-} = \frac{1}{\omega}. \quad (\text{A.19})$$

Note that the eigenvalues λ_{\pm} are θ -dependent, so the $0 < \theta < \pi$ distribution differs from the one at $\theta = 0$ not just by a rigid rotation.

Specifically for $A_R = 1/2$, i.e. $\alpha = i\pi \pmod{i2\pi}$, a second real solution with $E_0 = 0$ exists:

$$\varphi_0(x, y) = \exp\{-S(x + iy) - S^*(x - iy)\}, \quad A_R = 1/2. \quad (\text{A.20})$$

The solutions (A.17), (A.20) are not in $L^1(\mathbb{R}^2)$ while (A.18) is integrable for all $0 \leq \theta < \pi$, provided

$$\theta \neq \alpha \in \mathbb{R}, \quad \text{or} \quad 4\pi n - \theta < i\alpha < 4\pi n + \theta, \quad n \in \mathbb{N}_0. \quad (\text{A.21})$$

Likewise the constraint (3.5) is satisfied for (A.17) only with a divergent constant of proportionality, while for (A.18) it is satisfied with a finite one. The special solution (A.20) satisfies (3.5) with a finite constant of proportionality only if formally $\pi < \theta < 2\pi$ is assumed. This leaves only (A.18) as an acceptable $L^1(\mathbb{R}^2)$ solution. The average $\langle \mathcal{O} \rangle_{\varphi_0}$ computed from (A.16), (A.18) with $\mathcal{O}(z) = \exp jz$ correctly evaluates to

$$\langle e^{jz} \rangle_{\varphi_0} = \exp \left\{ \frac{e^{i\theta/2}}{4\omega} j^2 \right\} = \langle e^{jx} \rangle_{\Omega_0}, \quad (\text{A.22})$$

where $\Omega_0(x) = e^{-S(x)}$ is the ground state of \mathbf{P} . As a final test we also computed the equilibrium distribution of the Langevin dynamics numerically and found excellent agreement with the analytical formula for φ_0 , see Figure 7.

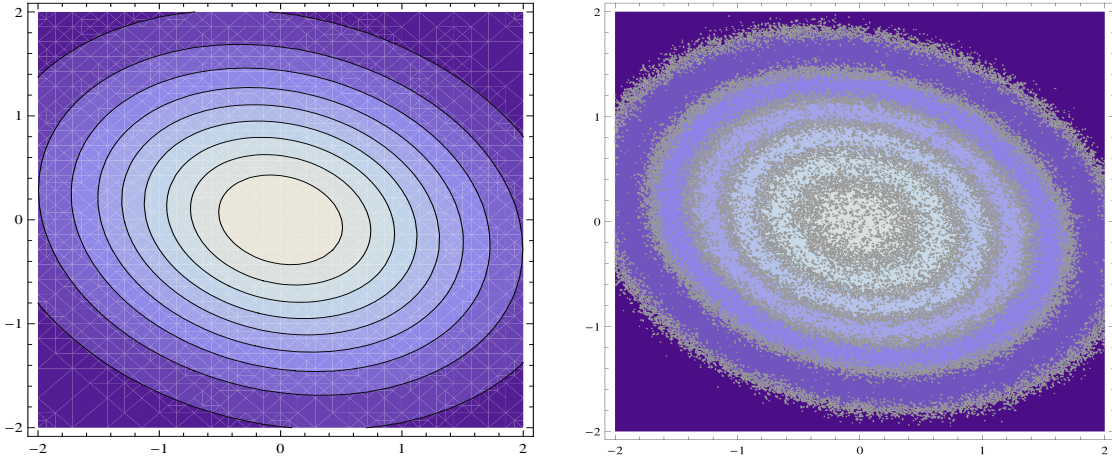


Figure 7: Ground state wave function for \mathbb{P} with parameters $A_R = 2, A_I = 1, \omega = 1, \theta = \pi/2$. Left: contour plot of analytical φ_0 from (A.16), (A.18). Right: results from complex Langevin simulations.

B: Resummation as a quantum mechanical ‘supertask’

The recursion relation (3.23) for the moments is virtually identical to a system of differential equations devised by J. Norton in the context of quantum mechanical ‘supertasks’ [25]. This gives rise to an amusing parallelism which we cannot resist mentioning. In brief, Norton considers a Schrödinger equation $i\partial_t\psi = H\psi$, where $\psi(t)$ is expanded with respect to some L^2 orthonormal basis

$$\psi(t) = \sum_{n \geq 0} f_n(t) e^{it} \varphi_n. \quad (\text{B.1})$$

The hamiltonian H is tri-diagonal in the basis $(\varphi_n)_{n \geq 0}$ and such that the f_n obey the recursion relation

$$f_n = \frac{a_{n-2}}{a_{n-1}} f_{n-2} - \frac{1}{a_{n-1}} f'_{n-1}, \quad n \geq 2, \quad (\text{B.2})$$

with $f_0 = \text{const}$ and $f_1(t)$ freely specifiable. The positive numbers a_n , $n \geq 0$, parameterize the hamiltonian, see Eqs. (13') and (17'') in [25]. In the $n = 2$ equation in (B.2) we allowed $a_0 > 0$ for later convenience; compared to $a_0 = 0$ this modifies some aspects related to normalizability but does not affect the overall structure. The term ‘supertask’ derives from the fact that for suitable choices of a_n and initial conditions the system can undergo spontaneous excitations with a pattern familiar from classical supertasks. That is, each excitation is aroused by a faster excitation of higher numbered excitations, ad infinitum, leading to severe violations of determinism. This can be achieved without sacrificing quantum mechanical principles, even in cases where the temporally conserved norm $\|\psi\|$ is finite. The construction capitalizes on the fact that the differential form of the time evolution law as set by the Schrödinger equation $i\partial_t\psi = H\psi$ may under-determine the time evolution for systems with an infinite dimensional Hilbert space. It is only through the subtle injection of additional pieces of information in terms of boundary conditions and domains that the associated propagation kernel, $t \mapsto e^{itH}$, if well-defined, fully determines the time evolution. The countable infinity underlying (B.2) and the corresponding basis $(\varphi_n)_{n \geq 0}$ is essential in this context – any truncation of (B.2) at some $n \leq N$ would lead to a closed system of differential equations which fully determines all functions f_n , $0 \leq n \leq N$, including f_1 . It is therefore a ‘supertask’ to borrow information from infinity to specify $f_1(t)$ by some principle.

The case where the a_n grow without bound is referred to as an “accelerated supertask”. In this situation the Schrödinger equation under-determines the time evolution and a wave function with normalizable initial condition will in general instantaneously evolve into a nonnormalizable one. For example with $f_n(0) = \delta_{n,0}$ all $f_n(t)$ are nonzero for arbitrarily small $t > 0$, with

$$f_n(t) = \frac{1}{n!} a_0 a_1 \dots a_{n-1} t^n + O(t^{n-2}), \quad n \geq 1. \quad (\text{B.3})$$

For a_n growing faster than n this suggests that $\sum_{n \geq 0} f_n(t)^2$ will diverge for arbitrarily small $t > 0$, even if (B.3) only provides the leading terms of an asymptotic expansion. If the $f_n(t)$ converge for $t \rightarrow \infty$, the limiting values are dictated by (B.1)

$$f_n(\infty) = \begin{cases} \frac{a_0 a_2 \dots a_{n-2}}{a_1 a_3 \dots a_{n-1}} f_0 & n \text{ even}, \\ \frac{a_1 a_3 \dots a_{n-2}}{a_2 a_4 \dots a_{n-1}} f_1(\infty) & n \text{ odd}. \end{cases} \quad (\text{B.4})$$

In simple cases like $a_0 = 0, a_n = a, n \geq 1$, normalizability of the wave function can be used as a criterion to restore uniqueness of the time evolution [25]. In the situation (B.3), (B.4) above normalizability is necessarily violated when $a_0 f_1(\infty) \neq 0$. To see this note that

$$\|\psi\|^2 := \sum_{n \geq 0} f_n(t)^2 - 2a_0 f_0 \int_0^t ds f_1(s), \quad (\text{B.5})$$

is formally conserved on account of (B.2). The rearranging of the sums is legitimate only if $\sum_{n \geq 1} a_n f_n(t) f_{n+1}(t)$ converges absolutely for all t . In the $t \rightarrow \infty$ limit the latter sum contains denumerably many identical terms of the form $f_0 a_0 f_1(\infty)$; so with $f_0 \neq 0$ the vanishing of $a_0 f_1(\infty)$ is a necessary condition for normalizability. In the following we are interested in situations where $a_0 f_1(\infty) \neq 0$ and argue that, instead, Borel summability can be used as a criterion to restore uniqueness of the time evolution. To this end we make contact with the results of Section 3.2.

Consider specifically the case

$$a_n = \sqrt{2n(2n+1)(2n+2)}, \quad n \geq 1, \quad (\text{B.6})$$

and set $m_0 = 1$ and

$$m_p(t) = (4\alpha)^{-p/4} \frac{\sqrt{2p(p-1)}!!}{a_0 f_0} f_{p/2}(\sqrt{4\alpha}t), \quad p = 2 \pmod{2}. \quad (\text{B.7})$$

Then the recursion relation (3.23) for the moments is mapped into (B.2) with the specific a_n in (B.6) and $f_0 = 1$. In particular (B.3) is mapped into $m_p(t) = \frac{p!}{(p/2)!} t^{p/2} + O(t^{p/2-1})$, as required, and the asymptotics (B.4) is mapped onto (3.33). As seen in Section 3.2 the formal power series in t obtained from (3.23) is asymptotic to the exact result for short times. It is also Borel summable and the Borel transform uniquely defines an $m_2(t)$ for all times with a nonzero asymptotics $m_2(\infty)$. Hence all moments $m_p(t)$ are uniquely determined and as shown in Section 3 they also coincide with the ones defined by the complex propagation kernel via (2.1).

Hence the Borel transform in this case augments precisely the piece of information otherwise supplied by the construction of the propagation kernel, the latter however cannot be achieved explicitly even in the simple case of a sextic anharmonic oscillator. The failed equivalence of (2.1) to the $\langle z^p \rangle_{R_t}$ moments highlights that a differently constructed semi-group may define a different time evolution with a different $t \rightarrow \infty$ asymptotics.

Although (B.6), (B.7) provides a mathematical isomorphism the conceptual interpretation of the m_p and the f_n is of course different. The m_p are averages of the observables x^p and are independent of the choice of basis in the underlying state space. Their dynamics (3.23) arises from instances of the observable flow equation (3.17) with respect to the stochastic time t . The dynamics of each fixed observable is driven by the quartic action via $\mathbf{L} = -e^{S/2} \mathbf{H} e^{-S/2} = \partial_x^2 - \partial_x S \partial_x$, and only when a complete set of observables is considered is the effective dynamics of the set governed by the tri-diagonal hamiltonian underlying (B.2). The f_n in (B.1), on the other hand, are the coefficients of the time dependent Schrödinger wave function with respect to a preferred basis, and as such do not directly qualify as observables. The time variable t refers to the physical time. Nevertheless the correspondence (B.7) cuts both ways. It could be used to design examples where the quantum mechanical supertasks can be solved by Borel transform. Conversely the underdetermination in the Schrödinger dynamics (3.23) is ultimately the reason for the failure of the Parisi-Klauder conjecture in the present context.

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